BULLETIN

of the

CALCUTTA MATHEMATICAL SOCIETY

VOLUME 47

1955

PUBLISHED BY THE CALCUTTA MATHEMATICAL SOCIETY
92, Upper Circular Road, Calcutta-9
1955

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SOLUTION OF PROBLEMS BY THE METHOD OF ANALOGUE *

By

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A mathematician engaged in the solution of a problem usually considers his work finished when his equations are fomulated and solved. Often, however, the numerical computation of the solution and its interpretation present as much difficulties as the steps leading to the solution.

Solution of problems by the method of analogue, whenever a suitable one can be devised, makes the interpretation of results comparatively easy. A well-known case is the torsion problem relating to a cylinder or a prism of any shape; such problems are solved by introducing two functions φ and ψ which are mathematically identical with the velocity-potential and stream-function of the motion of an incompressible fluid contained in a vessel of the same shape as the cylinder or prism which would be set up by rotating the vessel about its axis with an angular velocity equal to -1. For many problems involving electro-magnetic rotations we have neat hydrodynamical analogy. For instance, the case of an electric current passing radially from an axial wire, through a conducting metallic cylinder, in a uniform magnetic field, is analogous to the case of a rotating fluid under the action of suitable extraneous forces. The hydrodynamic analogue to problems relating to shearing of thin plate lamina is also well-known.

The main object in solving problems by the method of analogy is to make the physical interpretation of the analytical results comparatively easy or to devise experiments by which all the properties could be measured and plotted. An example of "analogy" experimental device is that in which a heat-flow or electric field problem is studied by means of measurements made on a flowing viscous fluid.

In making measurements by the method of analogy, considerable skill and ingenuity have been brought into play. The many analysing machines that have been devised are in most cases all illustrations of solutions of problems by the method of analogy. These have been classified as follows.

I. Machines for mathematical operations	Arithmetic	Keyboard computing machines. Punched card machines. Slide Rules.
	Trigonometric	Various Surveying Machines. Nomograms.
	Integrators	Planimeters. Harmonic Analysers. Optical integrators.
	Differentiation	Many simple devices of limited precision.

* An address delivered at the Annual Meeting of the Calcutta Mathematical Society in January, 1955.

II.	Machines for special purposes combining measurements with mathematical operations	Spectrophotometry	Colour Analyser with integrating mechanism (such as Hardy's Colour Analyser)
		Analysis of Spectra	Recording Comparator, with scale correction means. Machine for evaluating arithmetical differences.

(There are many instruments coming under this class).

III.	Machines for solving equations.	Algebraic Simultaneous linear	Electric circuit devices. Network analyser. Mechanical Linkages.
		Higher Degree	Devices employing Kelvin Balance. Electric Circuit Devices.
		Ordinary Differential	Differential Analyser.
		Partial Differential	Devices for special equations operating by analogy. Electronic computing machine.
		Integral.	Cinema integraph, for special equations and by successive approximations.

The great development of analysing machines in recent years has been due to the invention of different types of thermionic valves, photoelectric cells and thermistors and the introduction of devices leading to manufacture of high precision machines. With the modern electronic machines, the solutions of differential equations with appropriate boundary conditions, which when attempted unaided would take a number of days, can be completed in 2 or 3 hours. Richardson formulated the differential equations (hydrodynamical and thermodynamical) which would determine the atmospheric motions over a country. Their numerical solutions by the method of finite differences after taking due account of the initial conditions and boundary conditions proved to be a formidable task. But in some countries, the electronic machines are now made to solve these in 2 or 3 hours and to make a weather prediction with success.

The Network Analyser has had conspicuous success in solving simultaneous algebraic equations involving complex coefficients. This electric circuit device is a combination of adjustable resistances, inductances and capacitances, which may be interconnected in any desired manner and to which may be applied at a number of points alternating voltages which are adjustable in magnitude and phase. It is ordinarily used for the solution of electric power-system problems by the artifice of completely reproducing in miniature the essential parameters and interconnections of the system. In this operation, it is in essence solving simultaneous algebraic equations with complex coefficients but with certain limitations. The Network Analyser has also been used in solving the hydraulic problem of the flow of water in interconnected piping systems. Since the resistance of a pipe

is very nearly proportional to the current, it is necessary to make successive adjustments of the elements used in the net-work until this condition is satisfied in each branch. Hazen suggested some years ago a resistance network composed of tungsten lamps, for these satisfy the relationship over a considerable range. The combination of tungsten lamp with a thermionic tube for producing a current proportional to the voltage is thus capable of squaring a current. Such a combination has been utilised in a device for solving non-linear algebraic equations.

As is well-known there are several interesting designs for integrators. In cam mechanisms, there are usually three rotating shafts; one, whose angle of rotation is measured by u, controls the ratio; the second, whose angle of rotation is measured by v is the drive-shaft, and the third whose angle of rotation is measured by w, is the driven shaft. In such a mechanism

$$w = \int u dv.$$

The latest form of integrator consists of a photo-electric cell, which gives a response at any instant proportional to the total amount of light falling on its cathode. It is usually combined with an optical integrating sphere. If the light comes from a uniformly brilliant area, the response of the photoelectric cell gives an instantaneons measure of the area and hence of an integral. An adjustable cam may be employed to provide a functional control. The combination of functional control and an integrator can be used to evaluate special integrals. Thus a sinusoidal cam, or its equivalent and a variable-speed type of integrator constitute the essential components of an harmonic analyser.

Nearly eighty years ago, Sir William Thomson, indicated a method of solving differential equations by a mechanical interconnection of integrators. In the Massachusetts Institute of Technology, intensive work has been carried on the type of instrument known as "differential analysers". The differential gears are used to produce additions; a set of interconnected integrators is used to generate a desired function. Thus in using the analyser a variable coefficient may be introduced by generating it simultaneously in a portion of the machine set aside for the purpose. From this point of view of the machine, a differential equation with variable, but simply expressed coefficients, is transformed into a more complex equation with constant coefficients. Manual opertion in these machines has been dispensed with and an automatic following device has been introduced, using for the purpose a motor driven directly from a photo-cell through a thermionic tube amplifier.

Another machine also developed in the Massachusetts Institute of Technology is the Cinema Integraph. This is based on a suggestion made by Dr. Norbert Wiener that radiant energy could be used for the evaluation of integrals. If we have two parallel vertical screens with apertures, whose lower edges are stragiht lines (x-axis) and the upper edges are $y = F_1(x)$ and $y = F_2(x)$, and if the distance between the screens is large compared to the sizes of the apertures, and if further light from a linear incandescent filament is

made to pass through the screens, then it can be readily shown that light passing through the second screen is proportional to

$$\int F_1(x)F_2(x)\,dx.$$

If this light is gathered by a lens and impressed on a photocell, then the current in the cell becomes a measure of the integral. If the first screen is now shifted horizontally and parallely to the second, in accordance with a parameter λ, the variation of the photo-cell current will give a continuous evaluation of the integral with a cyclic kernel as a function of this parameter, that is,

$$\int F_1(\lambda - x)F_2(x)\,dx.$$

The light source is made as far as possible uniform, and for accurate recording, the positive portions of the integral are impressed on one photo-cell and the negative portions on a second, and the amount of light neceesary to be added to one of these through an adjustable shutter in order to produce a balance in the photo-cells is used as a measure. The balancing could not, of course, be done manually but the use of a photoelectric servomechanism enabled rapid and precise balancing to be obtained. To evaluate integrals with a general kernel,

$$\varphi(y) = \int K(x, y)f(y)\,dy,$$

for instance, Fourier transforms, in which

$$K(x, y) = A \sin xy,$$

one has to take a strip of motion picture film, and this will have on successive frames the successive sections K for increments in y, each section plotted as a function of x; for Fourier transforms, the frames will have sinusoidal plots of gradually increasing period in x.

It is now usual to classify computers into two families the "digital" and the "analogue". The digital computer, as distinguished from the 'analogue computer', is characterised by the fact that it does not measure, but it counts. The Hollerith Machine is a digital computing machine. The International Business Machine Corporation (IBM) which manufactures these machines, standardised the IBM card to a size of three and a quarter by seven and three-eighth inches, with its 80 columns of 12 punching positions each. This card is interchangeable among a variety of punching, sorting, tabulating, calculating and accounting machines, which deal with the cards mechanically, electrically and electronically. The same kind of card may hold the data of sampling surveys relating to income or occupation of a population, or some biological measurements, or agricultural measurements, or data of an accountant's audit or a corporation's income-tax, or data relating to meteorological observations or astronomical measurements. The punching of cards has to be made according to a specified number-code which represents the properties of the observed data and a stack of such cards forms its library or "memory". The cards

with punched holes speak just the kind of language which an electrical machine understands and with their assistance the data can be analysed in any manner we desire. An equally important advance was the teleprinter of the telegraph department, which transmits a message from one station to another by means of perforated tapes and a system of relays. These conceptions have played important roles in the development of modern electronic computers or electronic brain The ability to put command in code and to have them carried out by an electrical system is a major ingredient of the electronic brain.

The well-known "mechanical brain", built by Dr. Vannevar Bush, 30 years ago at M. I. T. was an analogue machine. But the latest types of electronic brains are digital machines This development involved a modification of the decimal system and the use of a hybrid, called "binary decimal", in which any two symbols, which are usually written as 0 and 1, are used. Every number can be represented by the combination of the two symbols. Thus, we get

Decimal notation	Binary notation	Decimal notation	Binary notation
0	0	3	11
1	1	5	101
2	10	6	110
4	100	7	111
8	1000	9	1001

The presence of one or the other of the symbols, 1 and 0, constitutes a binary digit. The phrase binary dig t has been, abbreviated in a new term "bit", meaning "a bit of information". In the binary system we need 10 bits to represent a decimal thousand, 20 bits to represent a million and 30 bits to represent a billion. It takes only one elctronic tube to represent a bit, and a total of 33 tubes to cover the range of the ten billion figure of an electroronic machine accumulator. In a machine 1 and 0 may be represented by any two physical effects; for instance, 1 may be represented by an electrical pulse and 0 by the absence of a pulse. As a magnetised tape passes under a coil, the presence or absence of a magnetised spot is converted into the presence or absence of an electrical signal, which in turn can be routed to an electronic tube. The simplification introduced by the binary system is that the addition of a digit simply means a reversal of the existing condition of the tube, that is, adding a pulse where there is none or wiping it out if there is one (that is, converting 1 to 0).

The first electronic computer, the Electronic Numerical Integrator and Computer know as ENIAC, was built during the war at the University of Pennsylvania for the Army Ordnance Department. It was later moved to the Ballistic Research Laboratory at the Aberdeen Proving Ground. It did not use the binary system but a "five-unit

code". It was a bulky instrument and used nearly 18000 electronic tubes. It had 20 accumulators, each accumulator consisting of 200 double-triode valves. These accumulators constituted the "electronic memory" and could store 20 numbers of 10 digits each. The accumulators function in a dynamic way, either sending out the number which it has been holding or receiving a new number and automatically adding to the one which it has been holding and all this is accomplished in a few microseconds.

The introduction of the binary system has led to a considerable compactness in the latest form of electronic computers. Instead of a perforated tape or punched card, the type of "memory" used in these machines is a magnetic tape, a photographic film, charged cathode-ray-tube surface, or simply a column of mercury in which numbers are stored in the dynamic form of waves moving at the speed of sound. The mercury ripple was a remarkable adjunct to the war time radar and it has proved to be an equally remarkable adjunct to the electronic computer. This is used to introduce delay in transmission of signals, the exact period of delay can be altered by changing the length of the mercury column and its temperature. The action is as follows. An electrical pulse or a signal on reaching a quartz crystal causes it to expand due to the piezoelectric effect; the crystal pushes aganist the mercury and a ripple runs along the column, which on reaching the end excites a second crystal and generates a new electrical impulse. This is fed back to the front end of the column by an amplifier circuit and this procees goes on until at the desired moment an electronic gate opens in the amplifier to switch the signal into some other circuit. Devices like this have been used in building up the complicated electronic computing circuits.

While remarkable skill and ingenuity have been brought into play in building up the different types of computing machines, they are all general purpose instruments. There are, however, many problems the solutions of which by the method of analogue are of great value. This enables one to design instruments to subject the solutions to experimental verification. These are all special purpose instruments. There is a vast field of work along this line. I will give an example from a problem in heat transfer.

This problem refers to the parallel flow and counter-flow heat exchangers. For parallel flow heat exchanger, a hydraulic analogue has been devised. This consists of two vertical glass tubes of uniform cross-section connected at the bottom by a capillary. The height of liquid in a tube at a given instant corresponds to the temperature of a stream of fluid at a certain point in the exchanger and time corresponds to length, that is, to heat transfer surface. The difference in temperatures between the two streams, which is the driving force for heat transfer between them has its analogue in the difference in heights of liquid in the corresponding tubes, which is the driving force for flow of liquid through the capillary. The heat transfer equations for the insulated parallel flow exchanger are

$$-C_1G_1d\theta_1 = \alpha K(\theta_1-\theta_2)\,dl = C_2G_2d\theta_2, \qquad (1)$$

where l denotes length along exchanger, α denotes the heat transfer surface per unit length of exchanger, C the isobaric heat capacity of fluid stream, K the overall heat

transfer coefficient per unit heat transfer surface, θ the temperature of fluid stream and the suffixes 1 and 2 refer to the hotter and the colder stream respectively.

The corresponding equations for the hydraulic analogue are

$$-S_1 dz_1 = F(z_1 - z_2)\, dt = S_2 dz_2. \tag{2}$$

Here z_1 and z_2 represent the heights of the liquid in the two tubes, S_1 and S_2 their cross-sections, F the condutance of the capillary and t the time. In order that equation (2) may be valid, the capillary must be sufficiently long and fine so that the flow through it is laminar and the acceleration effects are negligible. To obtain the relationship between temperature and height and between length and time, we write

$$\theta = n_\theta z \text{ and } l = n_l t.$$

Substituting these values in (1), we see that n_θ cancels out. Therefore, we can use any convenient value of this factor. Equation (2), coupled with equation (1), gives

$$n_l = (F/\alpha K)(C_1 G_1/S_1) = (F/\alpha K)(C_2 G_2/S_2) \tag{3}$$

or

$$S_2/S_1 = C_2 G_2 / C_1 G_1. \tag{4}$$

Thus F must be proportional to αK and S to CG if the analogy between fluid flow and heat transfer is to be valid. For simplicity let us first assume that the C's and K are constants so that S's and F are constants. The integral of equation (1) over the entire length l of the exchanger is

$$C_1 G_1(\theta_{1,l} - \theta_{1,o}) = C_2 G_2(\theta_{2,o} - \theta_{2,l}) \tag{5}$$

and similarly for equation (2),

$$S_1(Z_{1,l} - Z_{1,o}) = S_2(Z_{2,o} - Z_{2,l}) \tag{6}$$

Therefore, if $C_1 G_1 < C_2 G_2$, so that $S_1 < S_2$, then $\Delta\theta_1 > \Delta\theta_2$ and $\Delta Z_1 > \Delta Z_2$. In the opposite case we have merely to reverse the relative heights of liquid in the tubes and reverse their numbering. In an experiment, we set the initial difference of heights in the tubes to correspond to the initial difference of temperatures at the left end of the exchanger. On opening the valve in the line joining the tubes, we measure the time necessary to attain the difference of heights corresponding to the difference of temperatures at the other end of the exchanger. From the measured value of F/S_1 or F/S_2 and the calculated value of $C_1 G_1/\alpha K$ or $C_2 G_2/\alpha K$, we find n_l from equation (3) and consequently the necessary length of the exchanger from the relation $l = n_l t$, Alternatively, with an exchanger of given lenth, we can determine the value of K, the overall heat transfer coefficient.

A hydraulic analogue can also be designed for counter-flow heat exchanger. Such hydraulic analogue is essentially a differential analyser, which solves the system of

differential equations of heat transfer. We can also design electrical net work to find solutions for problems of this nature.

References.

Borden. P. A. (1929), Applications of the electric balance to the continuous solution of mathematical formulae. *Instrument*, **5**, (2)

Bush. V. (1931), *Journ Franklin Inst*, **212**.

——— (1934), *Journ. Franklin Inst* **217**.

Gould, K E (1928), A new machine for integrating a functional product *Journ. Math. and Phys of the M I T.*, **3**.

Gray, T S. (1931), A photo electric integraph *Journ. Franklin Inst.*, **212**.

Hazen, H L, Schurig O. R. Gardner, M F. (1930), The M. I T. network analyser, Trans. A. I E. E., **49**

Mallock, R R. M. (1933), An electrical calculating machine. *Proc. Roy. Soc*,, **140**.

Richardson, L F. (1922), *Weather Prediction by Numerical Process*, Camb. Univ. Press.

Standart, G. (1954), Hydraulic Analog for studying steady state heat-exchangers. *Collection Czechoslov. Chem. Commun*, **19**.

Thomson, Sir William, (1870), Mechanical integration of the general linear differential equation with variable coefficients *Proc. Roy. Soc.*, **24**. [Thomson and Tait, Natural Philosophy, **1**. p. 500]

GENERALIZED PERIODIC SINGULAR POINTS WITH APPLICATIONS TO FLOW PROBLEMS.

BY

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(*Communicated by Prof. B. R. Seth—Received July 2, 1953*)

Introduction. It is customary to think of singular points as the limiting cases of spherical or circular elements. Seth (1954) has shown that a generalization of this concept as ultimate forms of a family of closed surface bodies can be used to discuss the boundary value problems relating to these surfaces in exactly the same manner in which spherical and circular points are used to discuss the problems with spherical and circular boundaries respectively. He has solved the problem of motion of an ellipsoid and elliptic cylinder in a viscous fluid by applying the concept of ellipsoidal and elliptical doublets.

The singular points introduced by Seth are steady in character. This idea has been extended in this paper to include non-steady and periodic singular points. It is found that generalized periodic singular points can be used to discuss the vibration of a body in viscous fluid, by superposing on a solution due to a generalized periodic doublet, a solution due to a concentrated periodic force. Moreover, the forces operating can at once be correleted to the drag suffered by the body in the course of its vibration.

To define the generalized periodic source and doublet we proceed as follows:

Let V be the potential at an external point due to the solid A whose mass is M. Then by Gauss's theorem the normal flux across any surface enclosing S is

$$\iint_S -\frac{\partial V}{\partial n}\, ds = -4\pi M$$

the element of normal δn being drawn outwards. The limiting form of A when it reduces to a point or a line will be called the generalized source of strength M. Its potential will be taken as V. V satisfies the following conditions:—

(*i*) $\nabla^2 V = 0$, throughout all space occupied by the matter.

(*ii*) The flow across any closed surface containing it is constant.

(*iii*) GradV vanishes at infinity. The generalized source thus satisfies all the requisite conditions.

A doublet in the x- direction is obtained by displacing the source or solid through a small distance in the x- direction. The corresponding potential is $\mu\, \partial V/\partial x$, μ being the strength of the doublet.

In this paper we have discussed the vibration of a sphere, an infinite circular cylinder and an infinite elliptic cylinder in viscous fluid. The corresponding sources and doublets in the cases of sphere and circular cylinder are well known.

In the case of an elliptic cylinder we have V, the potential for a periodic elliptic source as

$$V = -\tfrac{1}{4}\pi\varrho\, ab\, (e^{-2\xi}\cos 2\eta + 2\xi)\, e^{int}$$

and for the corresponding doublet the potential φ is given as

$$\varphi = -\tfrac{1}{4}\pi\varrho\, ab\, \frac{\partial}{\partial x}(e^{-2\xi}\cos 2\eta + 2\xi)\, e^{int}$$

I. Vibration of a sphere

The linearised equations of motion are given as

$$-(\partial p/\partial x, \partial p/\partial y, \partial p/\partial z) + \mu\nabla^2(u, v, w) + \varrho(x, y, z) = \varrho(\partial u/\partial t, \partial v/\partial t, \partial w/\partial t) \tag{1}$$

and

$$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0. \tag{2}$$

On the surface of the sphere we have

$$u = U_0 e^{int}, v = 0, w = 0, \tag{3}$$

where n corresponds to the frequency of non-damped vibration, and at infinity

$$u = 0, \quad v = 0, \quad w = 0 \tag{4}$$

The usual method of attack consists in finding a solution of these equations with $p = p_0$, a constant. We propose to call it, 'the viscous counter-part' of the solution because it will not give rise to any pressure drag. A second solution of these equations is obtained by setting

$$u = -\partial\varphi/\partial x, \ v = -\partial\varphi/\partial y, \ w = -\partial\phi/\partial z \text{ and } p = \varrho\partial\varphi/\partial t$$

we propose to call it the potential counter-part, because it satisfies the Laplace's equation $\nabla^2\varphi = 0$. This will give rise to pressure drag only. The solution obtained by superposing these two satisfies all the boundary conditions.

It has not been pointed out by earlier workers that the non-linear terms due to potential counter-part need not be neglected in the equations of motion. Their presence does not alter the velocity at any point of the fluid, but simply modifies the pressure which is now given by

$$p/\varrho = \partial\varphi/\partial t - v^2/2$$

The drag suffered by the solid remains unaltered.

In the equations (1) we put $u = u_1 + u_2$, $v = v_1 + v_2$, $w = w_1 + w_2$ and $p = p_1 + p_2$, where (u_1, v_1, w_1) and p_1 are the velocity and pressure due to a periodic doublet and (u_2, v_2, w_2) and p_2 are the corresponding quantities due to a concentrated periodic force. The equations for p_1 and (u_1, v_1, w_1) are

$$-(\partial p_1/\partial x, \partial p_1/\partial y, \partial p_1/\partial z) + \mu\nabla^2(u_1, v_1, w_1) = \varrho(\partial u_1/\partial t, \partial v_1/\partial t, \partial w_1/\partial t), \tag{5}$$

and

$$\partial u_1/\partial x + \partial v_1/\partial y + \partial w_1/\partial z = 0. \tag{6}$$

The solution of these equations is obtained by setting

$$u_1 = -\partial\varphi/\partial x, \quad v_1 = -\partial\varphi/\partial y, \quad w_1 = -\partial\varphi/\partial z, \quad p_1 = \varrho\partial\varphi/\partial t$$

where $\nabla^2\varphi = 0$. But the velocity potential for a spherical periodic doublet is given by

$$\varphi = A\partial/\partial x(1/r)e^{int},$$

so that

$$\left.\begin{aligned} u_1 &= A\left(\frac{1}{r^3} - \frac{3x^2}{r^5}\right)e^{int}, \\ v_1 &= -\frac{3Axy}{r^5}\,e^{int}, \\ w_1 &= -\frac{3Axz}{r^5}\,e^{int}, \\ p_1 &= -A\varrho i n\frac{x}{r^3}\,e^{int}. \end{aligned}\right\} \tag{7}$$

The equations for (u_2, v_2, w_2) and p_2 are

$$-(\partial p_2/\partial x, \partial p_2/\partial y, \partial p_2/\partial z) + \mu\nabla^2(u_2, v_2, w_2) + \varrho(X, Y, Z) = (\partial u_2/\partial t, \partial v_2/\partial t, \partial w_2/\partial t) \tag{8}$$

and

$$\partial u_2/\partial x + \partial v_2/\partial y + w\partial_2/\partial z = 0 \tag{9}$$

We express the velocities by means of a vector potential (F, G, H) in the following menner

$$u_2 = \partial H/\partial y - \partial G/\partial z,$$
$$v_2 = \partial F/\partial z - \partial H/\partial x,$$
$$w_2 = \partial G/\partial x - \partial F/\partial y,$$

and the body forces in the like manner by means of a scalar potential Φ and a vector potential (L, M, N) by the formulae of the type

$$\left.\begin{aligned} X &= \partial\Phi/\partial x + \partial N/\partial y - \partial M/\partial z, \\ Y &= \partial\Phi/\partial y + \partial L/\partial z - \partial N/\partial x, \\ Z &= \partial\Phi/\partial z + \partial G/\partial x - \partial F/\partial y. \end{aligned}\right\} \tag{10}$$

The equations (8) can be written in such forms as

$$-\frac{\partial}{\partial x}p_2 + \mu\left\{\frac{\partial}{\partial y}(\nabla^2 H) - \frac{\partial}{\partial z}(\nabla^2 G)\right\} + \varrho\left(\frac{\partial\Phi}{\partial x} + \frac{\partial N}{\partial y} - \frac{\partial M}{\partial z}\right) = \varrho\left\{\frac{\partial}{\partial y}\frac{\partial H}{\partial t} - \frac{\partial}{\partial z}\frac{\partial G}{\partial t}\right\},$$

and particular solutions can be obtained by writing down solutions of the equations

$$\left.\begin{aligned} p_2 = \varrho\Phi,\ \mu\nabla^2 F + \varrho L &= \varrho\frac{\partial F}{\partial t}, \\ \mu\nabla^2 G + \varrho M = \varrho\frac{\partial G}{\partial t},\ \mu\nabla^2 H + \varrho N &= \varrho\frac{\partial H}{\partial t}. \end{aligned}\right\} \tag{12}$$

To solve the equations (12) we take one of them, *viz.*

$$\mu\nabla^2 H + \varrho N = \varrho\partial H/\partial t.$$

Since Φ, L, M, N, F, G, H vary harmonically we may put $\Phi = \Phi_1 e^{int}$ etc. The equation (12) takes the form

$$\frac{1}{\beta^2}\nabla^2 H_1 - H_1 = -\frac{N_1}{in}, \tag{13}$$

where $$\beta = -(in/\nu)^{\frac{1}{2}}.$$

The complementary function is

$$H_1 = \frac{e^{\beta r}}{r},$$

or $$H = \frac{1}{r}e^{int+\beta r}.$$

The particular solutions of (12) can now be put in the form

$$\left.\begin{aligned} F &= \frac{\beta^2}{in4\pi}\iiint \frac{1}{r}L_1' e^{int+\beta r}\,dx'\,dy'\,dz' \\ G &= \frac{\beta^2}{in4\pi}\iiint \frac{1}{r}M_1' e^{int+\beta r}\,dx'\,dy'\,dz' \\ H &= \frac{\beta^2}{in4\pi}\iiint \frac{1}{r}N_1' e^{int+\beta r}\,dx'\,dy'\,dz'. \end{aligned}\right\} \tag{14}$$

(X, Y, Z) can be put in the form (10) by taking

$$\left.\begin{aligned} \Phi &= -\frac{1}{4\pi}\iiint\left(X'\frac{\partial r^{-1}}{\partial x} + Y'\frac{\partial r^{-1}}{\partial y} + Z'\frac{\partial r^{-1}}{\partial z}\right)dx'\,dy'\,dz' \\ L &= \frac{1}{4\pi}\iiint\left(Z'\frac{\partial r^{-1}}{\partial y} - Y'\frac{\partial r^{-1}}{\partial z}\right)dx'\,dy'\,dz' \\ M &= \frac{1}{4\pi}\iiint\left(X'\frac{\partial r^{-1}}{\partial z} - Z'\frac{\partial r^{-1}}{\partial x}\right)dx'\,dy'\,dz' \\ N_1 &= \frac{1}{4\pi}\iiint\left(Y'\frac{\partial r^{-1}}{\partial x} - X'\frac{\partial r^{-1}}{\partial y}\right)dx'\,dy'\,dz', \end{aligned}\right\} \tag{15}$$

dash denoting the value of the corresponding quantity at (x', y', z') within the finite volume T which is such that (X, Y, Z) are different from zero in that volume and vanish outside it. r is the distance of this point from (x, y, z) and the integration extends through T.

Taking the case of a single concentrated force (Love, 1927), $X_0 e^{int}$ acting at the origin in the direction of the x-axis, we have

$$\Phi_1' = -\frac{1}{4\pi\varrho}X_0\frac{\partial R^{-1}}{\partial x'},$$

$$L_1' = 0,$$

$$M_1' = \frac{1}{4\pi\varrho}X_0\frac{\partial R^{-1}}{\partial z'},$$

$$N_1' = -\frac{1}{4\pi\varrho} X_0 \frac{\partial R^{-1}}{\partial y'},$$

where R denotes the distance of (x', y', z') from the origin. We may partition the space round the point (x, y, z) into thin sheets by means of spherical surfaces having that point as the centre, and thus we may express the integrations in (14) in such forms as

$$\iiint \frac{1}{r} N_1' e^{int+\beta r}\, dx'\, dy'\, dz' = \int_0^\infty -\frac{1}{4\pi\varrho} X_0\, e^{int+\beta r} \frac{dr}{r} \iint \frac{\partial R^{-1}}{\partial y'}\, ds,$$

where ds denotes an element of surface of a sphere with centre at (x, y, z) and radius equal to r. Now $\iint \partial R^{-1}/\partial y'$ is equal to zero where the origin is inside S, and to $4\pi r^2 \partial r_0^{-1}/\partial y$ when the origin is outside S, r_0 denoting the distance of (x, y, z) from the origin. In the former case $r_0 < r$ and in the latter case $r_0 > r$. We may therefore replace the upper limit of integration with respect to r by r_0, and find

$$H = -\frac{\beta^2}{in4\pi\varrho} \frac{\partial r_0^{-1}}{\partial y} \int_0^{r_0} r\, X_0\, e^{int+\beta r}\, dr.$$

Having found H we have no further use for r that appears in the process, and we may write r instead of r_0, so that r now denotes the distance of (x, y, z) from the origin. Then we have

$$\left.\begin{aligned} H &= -\frac{\beta^2}{in4\pi\varrho} \frac{\partial r^{-1}}{\partial y} \int_0^r rX_0\, e^{int+\beta r}\, dr, \\ G &= \frac{\beta^2}{in4\pi\varrho} \frac{\partial r^{-1}}{\partial z} \int_0^r rX_0\, e^{int+\beta r}\, dr, \\ F &= 0 \end{aligned}\right\} \quad (16)$$

so that we have the components (u_2, v_2, w_2) and p_2 due to a concentrated periodic force $X_0\, e^{int}$ at the origin as

$$\left.\begin{aligned} u_2 &= -\frac{X_0 e^{int}}{in4\pi\varrho}\left[\left(\frac{3x^2}{r^5} - \frac{1}{r^3}\right)\left\{e^{\beta r}(1-\beta r) - 1\right\} + e^{\beta r}\beta^2\left(\frac{x^2}{r^3} - \frac{1}{r}\right)\right], \\ v_2 &= -\frac{X_0 e^{int}}{in4\pi\varrho}\left[\frac{3xy}{r^5}\left\{e^{\beta r}(1-\beta r) - 1\right\} + e^{\beta r}\beta^2\frac{xy}{r^3}\right], \\ w_2 &= -\frac{X_0 e^{int}}{in4\pi\varrho}\left[\frac{3xz}{r^5}\left\{e^{\beta r}(1-\beta r) - 1\right\} + e^{\beta r}\beta^2\frac{xz}{r^3}\right], \\ p_2 &= \frac{X_0}{4\pi}\frac{x}{r^3} e^{int}. \end{aligned}\right\} \quad (17)$$

By superposing the two solutions we get

$$\left.\begin{aligned}
u &= Ae^{int}\left(\frac{1}{r^3}-\frac{3x^2}{r^5}\right)-\frac{X_0 e^{int}}{in4\pi\varrho}\left[\left(\frac{3x^2}{r^5}-\frac{1}{r^3}\right)\left\{e^{\beta r}(1-\beta r)-1\right\}+e^{\beta r}\beta^2\left(\frac{x^2}{r^3}-\frac{1}{r}\right)\right],\\
v &= Ae^{int}\left(-\frac{3xy}{r^5}\right)-\frac{X_0 e^{int}}{in4\pi\varrho}\left[\frac{3xy}{r^5}\left\{e^{\beta r}(1-\beta r)-1\right\}+e^{\beta r}\beta^2\frac{xy}{r^3}\right],\\
w &= Ae^{int}\left(-\frac{3xy}{r^5}\right)-\frac{X_0 e^{int}}{in4\pi\varrho}\left[\frac{3xy}{r^5}\left\{e^{\beta r}(1-\beta r)-1\right\}+e^{\beta r}\beta^2\frac{xz}{r^3}\right],\\
p &= \left(\frac{X_0}{4\pi\varrho in}-A\right)\varrho in\frac{x}{r^3}e^{int}.
\end{aligned}\right\}\quad(18)$$

The above solutions have to satisfy the boundary conditions (3) and (4). Putting the conditions we find

$$A=\frac{3}{2}U_0R\beta^{-2}[e^{-\beta R}-1+\beta R-\tfrac{1}{3}\beta^2R^2],$$

$$X_0=6\pi\varrho.in\beta^{-2}U_0Re^{-\beta R}.$$

Calculation of Resistance. The drag suffered by the sphere consists of two parts, one, to maintain the periodic doublet and the second, the resultant of the system of stresses on the sphere caused by the motion due to the concentrated periodic force at the centre.

The doublet gives rise to pressure drag only, because the velocity components (u_1, v_1, w_1) have been derived from the velocity potential $\partial/\partial x\ (1/r)\ e^{int}$. This drag is

$$\begin{aligned}
D_1 &= -\int_0^{\pi} p_1\cos\theta.2\pi R^2\sin\theta\,d\theta\\
&=(4/3).\pi\mu\beta^2Ae^{int},
\end{aligned}$$

where θ is the angle between the normal n of the sphere at a point of the $x\,z$ plane and the x-axis.

The drag experienced by the sphere due to motion given by the concentrated periodic force is

$$D_2=2\pi R^2\int_0^{\pi}X_n\sin\theta\,d\theta \tag{20}$$

where
$$X_n=\frac{1}{R}(zX_z+xX_x)$$

$$X_z=\mu(\partial w_2/\partial x+\partial u_2/\partial z)$$

$$X_x=-p_2+2\mu\partial u_2/\partial x.$$

Therefore
$$D_2=\tfrac{2}{3}\beta^2\nu e^{\beta R}(-1+\beta R-\tfrac{1}{2}e^{-\beta R})\frac{X_0e^{int}}{in}. \tag{21}$$

As $$R \to 0, \quad D_2 \to -X_0 e^{int}, \tag{22}$$
where $X_0 e^{int}$ is the periodic force applied at the origin.

The total drag D is given by

$$D = 4\pi\mu U_0 R[-3/2 + (3/2)\beta R - (1/6)\beta^2 R^2]e^{int} \tag{23}$$

which agrees with the result given by Lamb (1924) and Lorentz (1927).

II. INFINITE CIRCULAR CYLINDER

The equations of motion in this case are obtained from the previous one by putting $w=0$. Proceeding as in the case of the sphere, the potential counterpart' will now be obtained from the two-dimensional circular periodic doublet

$$\varphi = -A\frac{\partial}{\partial x}\log r\, e^{int}$$

Then (u_1, v_1) and p_1 are given as

$$\left.\begin{aligned} u_1 &= -A\left(\frac{I}{r^2} - \frac{2y^2}{r^4}\right)e^{int}, \\ v_1 &= -A\frac{2xy}{r^4}e^{int}, \\ p_1 &= -\varrho in\frac{x}{r^2}e^{int}. \end{aligned}\right\} \tag{24}$$

The equation of motion in the 'viscous counterpart' are

$$-(\partial p_2/\partial x, \partial p_2/\partial y) + \mu\nabla_1^2(u_2, v_2) + \varrho(X, Y) = \varrho(\partial u_2/\partial t, \partial v_2/\partial t), \tag{25}$$

and $$\partial u_2/\partial x + \partial v_2/\partial y = 0. \tag{26}$$

We take

$$\left.\begin{aligned} X &= \partial\Phi/\partial x + \partial N/\partial y, & Y &= \partial\Phi/\partial y - \partial N/\partial x, \\ u_2 &= \partial H/\partial y, & v_2 &= -\partial H/\partial x. \end{aligned}\right\} \tag{27}$$

The equations (25) can be satisfied if Φ, N and H satisfy the equations

$$p_2 = \varrho\Phi, \tag{28a}$$

and $$\mu\nabla_1^2 H + \varrho N = \varrho\partial H/\partial t. \tag{28b}$$

Since Φ, N and H vary harmonically as e^{int} we put $\Phi = \Phi_1 e^{int}$ etc. The equation (28b) takes the form

$$in\ \varrho H_1 - \mu\nabla_1^2 H_1 = \varrho N_1. \tag{29}$$

In view of the condition at infinity the solution of the equation

$$\nabla_1^2 H_1 + (in/\nu)\, H_1 = 0, \tag{30}$$

or $$\frac{\partial^2 H}{\partial r^2} + \frac{1}{r}\frac{\partial H}{\partial r} - \frac{in}{\nu}H_1 = 0 \tag{31}$$

is $$H_1 = K_0((in/\nu)^{\frac{1}{2}}r)$$

The particular solution of (29) is

$$H_1 = \frac{1}{2\pi\nu}\int\int K_0((in/\nu)^{\frac{1}{2}}r)N'_1\,dx'\,dy'. \tag{32}$$

Taking a force $X = X_0e^{int}$ in the direction of x-axis only Φ_1 and N_1 can be put in the form

$$\left.\begin{aligned} \Phi_1 &= \frac{1}{2\pi}\int\int X_1'\frac{\partial}{\partial x}(\log r)\,dx'\,dy', \\ N_1 &= \frac{1}{\partial x}\int\int X_1'\frac{\partial}{\partial y}(\log r)\,dx'\,dy'. \end{aligned}\right\} \tag{33}$$

Let $$\varrho\int\int X_1'\,dx'\,dy' = X_0,$$

so that there is a concentrated force X_0e^{int} acting at the origin in the direction of the x-axis. Therefore

$$\Phi_1 = \frac{X_0}{2\pi\varrho}\frac{\partial}{\partial x'}\log R,\ N_1 = \frac{X_0}{2\pi\varrho}\frac{\partial}{\partial y'}(\log R) \tag{34}$$

where r, R and dashes have precisely the same meaning as in the case of the sphere.

Proceeding exactly as before lines we get,

$$\left.\begin{aligned} H &= \frac{X_0e^{int}}{2\pi\varrho}\frac{\partial}{\partial y}\log r\int_0^{\gamma/\nu^{\frac{1}{2}}} t'K_0((in)^{\frac{1}{2}}t')\,dt', \\ \Phi &= \frac{X_0e^{int}}{2\pi\varrho}\frac{x}{r^2}, \\ N &= \frac{X_0}{2\pi\varrho}\frac{y}{r^2}, \end{aligned}\right\} \tag{35}$$

which gives

$$\left.\begin{aligned} u_2 &= \frac{-X_0e^{int}}{2\pi i\alpha^2\mu}\left(\frac{1}{r^2}-\frac{2y^2}{r^4}\right)\{i^{\frac{1}{2}}\alpha r\,K_1(i^{\frac{1}{2}}\alpha r)-1\}+\frac{X_0e^{int}}{2\pi\mu}\frac{y^2}{r^2}K_0(i^{\frac{1}{2}}\alpha r), \\ v_2 &= \frac{-X_0e^{int}}{2\pi i\alpha^2\mu}\frac{2xy}{r^4}\{i^{\frac{1}{2}}\alpha r K_1(i^{\frac{1}{2}}\alpha r)-1\}-\frac{X_0e^{int}}{2\pi\mu}\frac{xy}{r^2}K_0(i^{\frac{1}{2}}\alpha r), \\ p_2 &= \frac{X_0}{2\pi}\frac{x}{r^2}e^{int} \quad \text{where } n/\nu = \alpha^2. \end{aligned}\right\} \tag{36}$$

Superposing this solution on the one due to a potential doublet, we get

$$u = \left[\left(-A+\frac{X_0}{2\pi i\alpha^2\mu}\right)\left(\frac{1}{r^2}-\frac{2y^2}{r^4}\right)-\frac{X_0}{2\pi i\alpha^2\mu}\left\{\left(\frac{1}{r^2}-\frac{2y^2}{r^4}\right)i^{\frac{1}{2}}\alpha rK_1(i^{\frac{1}{2}}\alpha r)\right.\right.$$
$$\left.\left.-\frac{i\alpha^2y^2}{r^2}K_0(i^{\frac{1}{2}}\alpha r)\right\}\right]e^{int},$$

$$v = \left[\left(-A + \frac{X_0}{2\pi i\alpha^2\mu}\right)\frac{2xy}{r^4} - \frac{X_0}{2\pi\mu i\alpha^2}\left\{\frac{2xy}{r^4} i^{\frac{1}{2}}\alpha r K_1(i^{\frac{1}{2}}\alpha r) + \frac{i\alpha^2 xy}{r^2}K_0(i^{\frac{1}{2}}\alpha r)\right\}\right]e^{int},$$

$$p = \left(A + \frac{X_0}{2\pi i\alpha^2\mu}\right)\frac{i\alpha^2\mu x}{r^2} e^{int}. \tag{37}$$

The boundary conditions give

$$A = -U_0\left[\frac{2i}{\alpha^2\{\text{ker}\,(a\alpha) + i\,\text{kei}\,(a\alpha)\}} + a^2 + \frac{2ia\{\text{ker}'\,(a\alpha) + i\,\text{kei}'\,(a\alpha)\}}{\alpha\{\text{ker}\,(a\alpha) + i\,\text{kei}\,(a\alpha)\}}\right],$$

$$X_0 = \frac{4\pi\mu U_0}{\{\text{ker}\,(a\alpha) + i\,\text{kei}\,(a\alpha)\}}$$ where a is the radius of the cylinder.

The values of the functions expressed in series are

$$\left.\begin{aligned}
\text{ker}\,(x) &= -\log(\tfrac{1}{2}\gamma x)\,\text{ber}\,x + \frac{\pi}{4}\,\text{bei}\,x - \frac{x^4}{2^2.4^2}(1+\tfrac{1}{2}) + \ldots\ldots,\\
\text{kei}\,(x) &= -\log(\tfrac{1}{2}\gamma x)\,\text{bei}\,x - \frac{\pi}{4}\,\text{ber}\,x - \frac{x^2}{2^2} - \frac{x^6}{2^2.4^2.6^2}(1+\tfrac{1}{2}+\tfrac{1}{3}) + \ldots\ldots,\\
\text{ber}\,(x) &= 1 - \frac{x^4}{2^2.4^2} + \ldots\ldots,\\
\text{bei}\,(x) &= \frac{x^2}{2^2} + \frac{x^6}{2^2.4^2.6^2} + \ldots\ldots,\quad \text{where } \gamma = \log(\text{Euler's const.})
\end{aligned}\right\} \tag{38}$$

Resistance. As in the case of the sphere let D_1 and D_2 be the drags suffered by the cylinder due to the motions given by the potential doublet and concentrated force respectively.

Then
$$D_1 = -a\int_0^{2\pi} p_1 \cos\theta\, d\theta$$
$$= A\pi\mu\, i a^2 e^{int}, \tag{39}$$

where θ is the angle between the radius vector and the x-axis.

$$D_2 = a\int_0^{2\pi}(p_{rx})_{r=a}\, d\theta$$

$$p_{rx} = -(p_2 + 2\mu\,\partial u/\partial x\cos\theta + \mu(\partial u/\partial y + \partial v/\partial x)\sin\theta.$$

Therefore
$$D_2 = -\frac{X_0 e^{int}}{2} + \frac{X_0 e^{int}}{2}a\alpha\{\text{ker}'\,(a\alpha) + i\,\text{kei}'\,(a\alpha)\}. \tag{40}$$

For small values of $a\alpha$

$$\left.\begin{aligned}
\text{ker}'\,(a\alpha) &= -\frac{1}{a\alpha} + \frac{\pi a\alpha}{8},\\
\text{kei}'\,(a\alpha) &= -\frac{a\alpha}{2}\log\tfrac{1}{2}(\gamma a\alpha) + \frac{a\alpha}{4}.
\end{aligned}\right\} \tag{41}$$

As the radius a of the cylinder tends to zero

$$\{\ker'(a\alpha)+i\,\mathrm{kei}'(a\alpha)\to-\frac{1}{a\alpha},$$

so that
$$D_2\to-X_0e^{int},$$
where $X_0\,e^{int}$ is the concentrated force applied at the origin.

Total drag D is given by

$$D=-\pi\mu U_0a^2\alpha^2\left[i-\frac{4}{a\alpha}\,\frac{\ker'(a\alpha)+i\,\mathrm{kei}'(a\alpha)}{\ker(a\alpha)+i\,\mathrm{kei}(a\alpha)}\right] \tag{43}$$

which agrees with the result given by M. Ray, (1933).

III. An Infinite Elliptic Cylinder

The equations of motion in this case are the same as in the previous one We take (u_1, v_1) as usual the velocity components due to a periodic doublet which in this case is

$$\varphi=A\frac{\partial}{\partial x}\{-\tfrac{1}{2}\pi\varrho ab(e^{-2\xi}\cos 2\eta+2\xi)\}e^{int}$$
$$=-D\,e^{-\xi}\cos\eta\,e^{int},$$

where A and D are constants.

So the stream function is

$$\psi_1=D\,e^{-\xi}\sin\eta\,e^{int},$$

which gives

$$\left.\begin{aligned} u_1&=-\partial\psi_1/\partial y,\\ v_1&=\partial\psi_1/\partial x,\\ p_1&=-i\varrho n\,De^{-\xi}\cos\eta\,e^{int}.\end{aligned}\right\} \tag{45}$$

To determine (u_2, v_2) and p_2 the velocity components and pressure given by the periodic concentrated force, we make exactly the same substitutions in the equations of motion as in the previous case of circular cylinder, getting thereby

$$\left.\begin{aligned}\Phi_1&=\frac{1}{2\pi}\int\!\!\int X_1'\pi\varrho\frac{ab}{2}\,\frac{\partial}{\partial x}(e^{-2\xi}\cos 2\eta+2\xi)dx'\,dy',\\ N_1&=\frac{1}{2\pi}\int\!\!\int X_1'\pi\varrho\frac{ab}{2}\,\frac{\partial}{\partial y}(e^{-2\xi}\cos 2\eta+2\xi)dx'\,dy'.\end{aligned}\right\} \tag{46}$$

Letting
$$\varrho\int\!\!\int X_1'\,dx'\,dy'=X_0,$$

$$\left.\begin{aligned}\Phi_1&=X_0\frac{ab}{c}\,e^{-\xi}\cos\varrho,\\ N_1&=X_0\frac{ab}{c}\,e^{-\xi}\sin\eta.\end{aligned}\right\} \tag{47}$$

Also
$$p_2 = \varrho\Phi, \tag{48}$$
$$\mu \nabla_1^2 H_1 - in\varrho H_1 = -\varrho N_1 \tag{49}$$

Using the transformation
$$(x + iy) = c \cosh(\xi + i\eta)$$
the equation (49) is transformed into
$$\frac{\partial^2 H_1}{\partial \xi^2} + \frac{\partial^2 H_1}{\partial \eta^2} - \tfrac{1}{2} i\alpha^2 c^2 (\cosh 2\xi - \cos 2\eta) H_1 = \frac{-\varrho}{2\mu} N_1 (\cosh 2\xi - \cos 2\eta), \tag{50}$$
where $\alpha^2 c^2 = n\varrho/\mu$

To find the complementary function of (50), viz, the solution of the equation
$$\frac{\partial^2 H_1}{\partial \xi^2} + \frac{\partial^2 H_1}{\partial \eta^2} - \tfrac{1}{2} i\alpha^2 c^2 (\cosh 2\xi - \cos 2\eta) H_1 = 0 \tag{51}$$

Let us put $H_1 = U(\xi)\, V(\eta)$ where $U(\xi)$ is a function of ξ and $V(\eta)$ is a function of η only. Then the equation (51) can be broken into
$$\frac{\partial^2 U}{\partial \xi^2} - (a + \tfrac{1}{2} i\alpha^2 c^2 \cosh 2\xi) U = 0, \tag{52a}$$
$$\frac{\partial^2 V}{\partial \eta^2} + (a + \tfrac{1}{2} i\alpha^2 c^2 \cos 2\eta) V = 0, \tag{52b}$$
where a is a separation constant.

The possible values of V are the Mathieu functions ce_0, ce_1, ce_2. se_1. se_2. corresponding to values a_0, a_1, a_2. b_1, b_2.. respectively, these values being expressible in terms of the other constant in the equation. An equation of the form (52a) has been studied by J Dougall (1915-16) who considered the equation
$$\frac{\partial^2 U}{\partial \xi^2} + (\tfrac{1}{2}\alpha^2 c^2 \cosh 2\xi - s^2) U = 0, \tag{53}$$
and found the solution valid for $\xi = \infty$ in the form
$$G(\nu, s, \alpha c, \xi) = \sum_{n=-\infty}^{\infty} a'_n J_n(\tfrac{1}{2}\alpha c e^{-\xi})\, G_{n+\nu}(\tfrac{1}{2}\alpha c e^{\xi}), \tag{54}$$
where
$$G_m(z) = (\pi/2 \sin m\pi)(J_{-m}(z) - \cos m\pi J_m(z))$$
and ν is determined in terms of s only by a certain equation and constants a_n have definite forms.

Now corresponding to ce_0, ce_3, se_2...it is found that $\nu = 0$ and that corresponding to ce_1, se_1, ce_3, $\nu = 1$. Also the constants a'_n have forms given by

$$a'_n = \Phi'(n+\tfrac{1}{2}\nu)/\Phi'(\tfrac{1}{2}\nu), \tag{55}$$

where if $\frac{1}{2}s = \gamma$ and $\frac{1}{4}\alpha c = \lambda$. The function $\Phi\ (z)$ is defined by

$$\Phi'(z) = \frac{\lambda^2 z}{\pi(z+\gamma)\,\pi(3-\gamma)}\{1-\lambda^4 A_z^{(1)}+\lambda^8 A_z^{(2)}\ldots\}, \tag{56}$$

where $A_z^{(1)}$, $A_z^{(2)}$, etc. are function of z and γ. The solution required in the present case for equation (52a) may be obtained from Dougall's solution by writing $i^{\frac{1}{2}}\alpha c$ in place of αc and will be denoted by

$$g_\nu(\xi) = \sum_{n=-0}^{\infty} a'_n I_n(\tfrac{1}{2}i^{\frac{1}{2}}\alpha ce^{-\xi})\ K_{n+\nu}(\tfrac{1}{2}i^{\frac{1}{2}}\alpha ce^{\xi}). \tag{57}$$

Accordingly a complete solution of the equation (51) may be written as

$$H_1 = (A_m ce_m + B_m se_m)\ g_m(\xi).. \tag{58}$$

For our purpose we take

$$H_1 = B_1 se_1 g_1(\xi), \tag{59}$$

corresponding to $\nu = 1$.

Now for small values of αc, se_1 is equal to $\sin\eta$ and to the same order of approximation $g_1(\xi)$ can be replaced by the single term $I_0(\frac{1}{2}i^{\frac{1}{2}}\alpha ce^{-\xi})\ K_1(\frac{1}{2}i^{\frac{1}{2}}\alpha ce^{\xi})$ in the neighbourhood of the cylinder, the corresponding a'_0 in this case being unity. All the other terms of the series when calculated contain αc, α^2c^2 etc. Whereas the term retained, namely that corresponding to $n = 0$ contains the term of order $1/\alpha c$. This justifies the neglect of all terms in $g_1(\xi)$ except the one for $n = 0$.

Having found the complementary function of the equation (50) we get the particular solution as in the last two cases in the form

$$H = \left[\frac{-X_0 e^{int}}{2\pi\mu i\alpha}\{I_0(\tfrac{1}{2}i^{\frac{1}{2}}\alpha ce^{-\xi})\ K_1(\tfrac{1}{2}i^{\frac{1}{2}}\alpha ce^{\xi})\} + \frac{X_0 e^{int}}{\pi\mu i\alpha^2 c}\,e^{-\xi}\right]\sin\eta, \tag{60}$$

it being supposed that the volume of the cylinder per unit length is unity *i.e.* $\pi\varrho ab = 1$.

The case of the circular cylinder may be deduced from here by keeping the semi-major axis a constant and letting $\xi\to\infty$. Thus $c\to 0$, $ce^{\xi}/2\to a$ and the ellipse tends to a circle of radius a and the above expression reduces to

$$\left(-\frac{X_0}{2\pi\mu i\alpha}\ K_1(i^{\frac{1}{2}}\alpha a) + \frac{X_0}{2\pi\mu i\alpha^2}\right)\sin\theta. \tag{61}$$

which agrees with the result obtained in the case of circular cylinder,

ψ_2, the stream function of the motion for a concentrated periodic force is

$$\psi_2 = -H$$

$$= \left(\frac{X_0}{2\pi\mu i\alpha} g_1(\xi) - \frac{X_0}{\pi\mu i\alpha^2 c} e^{-\xi} \right) \sin\eta \, e^{int} . \tag{62}$$

Superposing the two solutions we get

$$\psi = \psi_1 + \psi_2$$

$$= \left\{ \frac{X_0}{2\pi\mu i\alpha} g_1(\xi) + \left(D - \frac{X_0}{\pi\mu i\alpha^2 c} \right) e^{-\xi} \right\} \sin\eta e^{int} \tag{63}$$

$$= (Ag_1(\xi) + Be^{-\xi}) \sin\eta e^{int}, \tag{64}$$

where
$$A = X_0/(\pi\mu i\alpha^2 c),$$
$$B = D - X_0/(\pi\mu i\alpha^2 c).$$

Thus
$$u = -h^2 c[\cosh\xi \sin^2\eta\{Ag'_1(\xi) - Be^{-\xi}\} + \sinh^2\xi\cos^2\eta\{Ag_1'(\xi) + Be^{-\xi}\}]e^{int} \tag{65}$$

$$v = h^2 c \sin\eta \cos\eta[\sinh\xi\{Ag'_1(\xi) Be^{-\xi}\} - \cosh\xi\{Ag_1(\xi) + Be^{-\xi}\}]e^{int} \tag{66}$$

$$p = -i\mu\alpha^2 Be^{-\xi} \cos\eta e^{int} \text{ and } h^2 = 1/c^2(\sinh^2\xi + \sin^2\eta), \tag{67}$$

the dashes denote differentiation with respect to ξ.

The boundary conditions give

$$\left.\begin{aligned} X_0 &= \frac{-2\pi\mu i\alpha c U_0 e^{\xi_0}}{\{g_1(\xi_0) + g'_1(\xi_0)\}} \\ D &= \frac{U_0 c e^{\xi_0}\{-2/\alpha + \cosh\xi_0 g_1(\xi_0) - \sinh\xi_0 g'_1(\xi_0)}{\{g_1(\xi_0) + g'_1(\xi_0)\}} \end{aligned}\right\} \tag{68}$$

Resistance. The drag component given by the potential doublet is

$$D_1 = \int_0^{2\pi} -p_1 \sinh\xi_0 \cos\eta c d\eta$$

$$= i\mu\alpha^2 c\pi D e^{-\xi_0} \sinh\xi_0 e^{int} \tag{69}$$

taken round the transverse section.

D_2, the component due to the concentrated periodic force, is

$$D_2 = \int\left[\left(-p_2 + 2\mu\frac{\partial u_2}{\partial x}\right)\sinh\xi_0\cos\eta + \mu\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial x}\right)\cosh\xi_0 \sin\eta\right] c d\eta$$

$$= i\pi\mu\alpha^2 c e^{-\xi_0}\left\{\frac{-X_0}{\pi\mu i\alpha^2 c}\sinh\xi_0 - \frac{X_0}{2\pi\mu i\alpha} g_1(\xi_0)\cosh 2\xi_0\right\} e^{int} \tag{70}$$

with the order of approximation taken initially.

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Now as
$$\xi \to 0 \quad D_2 \to -X_0 e^{int} \tag{71}$$
where $X_0\, e^{int}$ is the concentrated force applied at the centre.

The total drag is given by

$$D = i\pi\mu\nu^2 c e^{-\xi_0}\left[\frac{U_0 c e^{\xi_0} \sinh \xi_0}{g_1(\xi_0) + g'_1(\xi_0)}\{\cosh \xi_0 g_1(\xi_0) - \sinh \xi_0 g'_1(\xi_0)\} + \frac{U_0 c e^{\xi_0} g_1(\xi_0)}{g_1(\xi_0) + g'_1(\xi_0)} \cosh 2\xi_0\right] \tag{72}$$

which in the value obtained by Ray (1936).

In conclusion I thank Prof B. R. Seth for his kind help in the preparation of this paper.

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References.

Dougall, J (1915-16), *Proc Edin. Math Soc*, **34**.
Seth, B R (1954), *Proc Ind. Acad. Sci.*, **40**, 25
Love, A. E H (1927), *Mathematical Theory of Elasticity*, 4th Edn, pp 303-304.
Lamb, H (1934), *Hydrodynamics*, 5th. Edn, p 609
Lorentz, (1927) *Lectures on Theoretical Physics*, vol I, p. 89.
Ray, M (1933), *Bull Cal. Math Soc*, **35**.
Ray, M (1936) *Z AMM*, **16**, 99.

A NOTE ON DETERMINANTS WITH BINOMIAL ELEMENTS.

BY

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(*Received—September 24, 1953.*)

1. Let G denote a matrix of the n^{th} order in which the $(i,j)^{th}$ element is

$$g_{ij} = a_{ij} + b_{ij}.$$

It is the object of this paper to obtain a canonical form to which $|G|$ reduces and further to extend the process so as to be applicable to any minor of G. We note (Rice. 1925) that $|G|$ is, in general, made up of 2^n determinants which are classified as follows:

(*i*) 1 determinant with n rows containing a's only.

(*ii*) ${}^nc_1$ determinants with $(n-1)$ rows containing a's and 1 row containing b's.

... ...

$(r+1)$. ${}^nc_r$ determinants with $(n-r)$ rows containing a's and r rows containing b's.

... ...

$(n+1)$. 1 determinant with n rows containing b's only.

2. Let us denote the matrix (a_{ij}) and (b_{ij}) by A and B respectively and the inverse of A ($|A| \neq 0$) by $A^{-1} = (a^{ij})$ so that $\sum_{k=1}^{n} a_{ik}a^{kj} = \delta_i^j$ Consider now the product matrix $C = BA^{-1}$. The $(i, j)^{th}$ element in this product is then expressed as

$$C_i^j = \sum_{k=1}^{n} b_{ik}a^{kj}. \tag{2}$$

Let I_r denote the sum of all the ${}^nc_r$ diagonal minor determinants of r^{th} order in the matrix C. Observing now that all the ${}^nc_r$ determinants in the $(r+1)^{th}$ class noted above, when multiplied by A^{-1} give only diagonal minor determinants of C of the r^{th} order, distinct from one another, it follows that the value of the product $|G||A^{-1}|$ must be $1 + I_1 + I_2 + \dots + I_n$. Hence

$$|g_{ij}| = (1 + I_1 + I_2 + \dots + I_n)\,|a_{ij}| \tag{3}$$

which is the required canonical expression.

If the matrix B is such that $|B| \neq 0$, we get an alternative form of (3) by interchanging a's and b's. This will introduce the matrix inverse to C. If, however, B is of rank r, the rank of C is r and the canonical form of $|G|$ will be given by

$$|g_{ij}| = (1 + I_1 + I_2 + \dots + I_r)\,|a_{ij}|. \tag{4}$$

Replacing b_{ij} in (1) by λb_{ij}, where λ is a parameter, we get

$$|a_{ij}+\lambda b_{ij}|=(1+\lambda I_1+\lambda^2 I_2+\ldots+\lambda^n I_n)\,|a_{ij}|. \tag{5}$$

Hence the determinantal equation $|a_{ij}+\lambda b_{ij}|=0$, is equivalent to

$$\lambda^n I_n+\lambda^{n-1}I_{n-1}+\ldots\lambda I_1+1=0. \tag{6}$$

3. Next we proceed to obtain a formula similar to (3) corresponding to a minor of $|G|$ derived by striking out a set of s successive rows $i_1, i_2,\ldots i_s$ and a set of successive columns $j_1, j_2\ldots j_s$. We observe that this minor of $(n-s)^{\text{th}}$ order is equivalent to a determinant of n^{th} order $|G'|$ derived from $|G|$ by replacing the elements of all the i-rows of $|G|$ by zeros except the s elements $(i_1, j_1)^{\text{th}}$, $(i_2, j_2)^{\text{th}}\ldots(i_s, j_s)^{\text{th}}$, each of which is replaced by 1. Let us call similarly-constructed minors of $|A|$ and $|B|$ by $|A'|$ and $|B'|$ respectively. Now denoting the $(n-s)$ remaining rows by $l_1, l_2,\ldots l_{n-s}$ in succession, $|G'|$ may be broken up into 2^{n-s} determinants of the n^{th} order in all of which the s i-rows are identical and same as those in $|G'|$, These determinants are classified in respect of the $(n-s)$ l-rows as follows:

(*i*) 1 determinant A' in which $(n-s)$ l-rows contain a's only

... ...

$(r+1)$. ${}^{(n-s)}c_r$ determinants in which $(n-s-r)$ l-rows contain a's and r l-rows contain b's.

... ...

$(n-s+1)$ 1 determinant B' in which $(n-s)$ l-rows contain b's only.

4. Consider now a single member of the $(r+1)^{\text{th}}$ class noted above and multiply it by $|A^{-1}|$. An i-row, for instance i_p, will contribute in the product a row i_p, containing elements $a^{j,1}, a^{j,2},\ldots a^{j,n}$ in successive columns. An l-row containing b's will contribute a corresponding l-row containing elements $c_l^1, c_l^2\ldots c_l^n$. An l-row containing a's, for instance l_p, will contribute an l_p-row of zeros except the diagonal element $(l_p, l_p)^{\text{th}}$, which is 1. Hence the product is a determinant of $(s+r)^{\text{th}}$ order. It is, however, easy to see that the totality of such ${}^{(n-s)}c_r$ determinants is represented in the sum of the ${}^{(n-s)}c_r$ diagonal minor determinants of $(s+r)^{th}$ order that can be formed from the matrix $C'=B'A^{-1}$ *by including always the s i-rows*. Let us denote this sum by I'_r. Then

$$|G'|\,|A^{-1}|=(I'_0+I'_1+\ldots I'_{n-s}),$$

whence

$$|G'|=(I'_0+I'_1+\ldots+I'_{n-s})\,|a_{ij}| \tag{7}$$

It must be noted here that I'_0 represents the diagonal minor of $|C'|$, in symbolic notation $\begin{pmatrix} i_1, i_2,\ldots i_s \\ i_1, i_2,\ldots i_s \end{pmatrix}$ and I'_{n-s} the determinant $|C'|$ itself.

5. In the recent unified field theory of Einstein the fundamental determinant $|G|$ of the non-symmetric metric tensor $g_{\mu\nu}$ yields a determinant with binomial elements when the metric tensor is split up into its symmetric and anti-symmetric parts. If B is symmetric and A anti-symmetric, then, since $n=4$ and $I_1=I_3=0$, the formula (3) reduces to

$$|g_{ij}|=(1+I_2+I_4).|a_{ij}|. \tag{8}$$

Similar simplification is possible also in respect of (7). It is found that the contravariant tensor $g^{\mu\nu}$ is given by the equation

$$|G|g^{\mu\nu}=|B|b^{\mu\nu}+|A|a^{\mu\nu}-(b^{\mu\alpha}a_{\alpha\beta}b^{\beta\nu})|B|-(a^{\mu\alpha}b_{\alpha\beta}a^{\beta\nu})|A| \tag{9}$$

where $$g^{\mu\nu}=\frac{\text{cofactor of } g_{\nu\mu}}{|G|}.$$

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Reference.

Rice, Lapire Hall, (1925). *Journ. Math. & Phys.*, 4, 62.

ON A GENERALISATION OF HERMITE'S POLYNOMIAL—I

By

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(*Received - August 18, 1953*)

1. **Introduction.** The object of the present paper is to generalise ordinary Hermite's polynomial $H_n(z)$ of integral order n and to study some of their properties. The generalised Hermite's polynomials $H_{km}(z)$ and $H_{km+1}(z)$ are defined as

$$H_{km}(z) = e^{z^k} D_{(k)}^{(km)} (e^{-z^k}) \tag{1.1}$$

and

$$H_{km+1}(z) = -e^{z^k} D_{(k)}^{(km+k-1)} (e^{-z^k}) \tag{1.2}$$

where (*i*) m and mk are *positive* integers (including zero)

but (*ii*) $k \neq 0$ (clearly k may also be *fractional*)

and the differential operators

(*iii*) $D_{(k)}^{(k)} \equiv \dfrac{d}{dz} \dfrac{1}{z^{k-2}} \dfrac{d}{dz}$, (*iv*) $D_{(k)}^{(k+1)} \equiv \dfrac{d}{dz} D_{(k)}^{(k)}$ so that

(*v*) $D_{(k)}^{(k-1)} \equiv \dfrac{1}{z^{k-2}} \dfrac{d}{dz}$ and (*vi*) $D_{(k)}^{(km)}$ means the operators $D_{(k)}^{(k)}$

operating on a function m times succesively.

Obviously for $k = 2$, the H_{km}-polynomials reduce to the familiar Hermite's polynomial $H_n(z)$.

Actually operating by $D_{(k)}^{(k)}$ on e^{-z^k} term-by-term m times successively, we get

$$D_{(k)}^{(km)} (e^{-z^k}) = (-)^m k^{2m} \frac{\Gamma(m+1/k)}{\Gamma(1/k)} \, {}_1F_1(m+1/k\,;\, 1/k\,;\, -z^k).$$

Applying Kummer's first transformation, *viz.*:

$${}_1F_1(\alpha\,;\, \varrho\,;\, \zeta) = e^{\zeta} {}_1F_1(\varrho-\alpha\,;\, \varrho\,;\, -\zeta)$$

we have therefore

$$H_{km}(z) = (-)^m k^{2m} \frac{\Gamma(m+1/k)}{\Gamma(1/k)} \, {}_1F_1(-m\,;\, 1/k\,;\, z^k). \tag{1.3}$$

Next from (1.2)

$$-e^{-z^k} H_{km+1}(z) = D_{(k)}^{(km+k-1)}(e^{-z^k}) = D_{(k)}^{(k-1)} D_{(k)}^{(km)} (e^{-z^k}) = z^{2-k} \frac{d}{dz}\{e^{-z^k} H_{km}(z)\} \tag{1.4}$$

so that by (1.3), we have

$$H_{km+1}(z) = (-)^m k^{2m+1} z \frac{\Gamma(m+1/k+1)}{\Gamma(1/k+1)} {}_1F_1(-m\,;\,1+1/k\,;\,z^k) \tag{1.5}$$

Now using the formula

$$(-)^r(-m)_r = \frac{m\,!}{(m-r)\,!} \tag{1.6}$$

we easily derive from (1.3) and (1.5) the following expansions:

$$H_{km}(z) = (-)^m k^{2m}\Gamma(m+1/k) \sum_{r=0}^{m} \binom{m}{m-r} \frac{(-)^r z^{kr}}{\Gamma(r+1/k)} \tag{1.7}$$

and

$$H_{km+1}(z) = (-)^m k^{2m+1}\Gamma(m+1/k+1) \sum_{r=0}^{m} \binom{m}{m-r} \frac{(-)^r z^{kr+1}}{\Gamma(r+1/k+1)} \tag{1.8}$$

2. Generating functions. Multiplying each term in the expansion of ${}_0F_1(1/k\,;\,-z^k h^k)$ by e^{-h^k} and then integrating term-by-term w. r. t. h over $(0, \infty)$, which is quite justified, we at first notice that

$$\int_0^\infty e^{-h^k}{}_0F_1(1/k\,;\,-z^k h^k)\,dh = \Gamma(1+1/k)e^{-z^k}$$

Then operating on both sides by $D_{(k)}^{(k)}$ m times successively, we get

$$(-)^m k^{2m} \int_0^\infty e^{-h^k} h^{km}{}_0F_1(1/k\,;\,-z^k h^k)\,dh$$

$$= \Gamma(1+1/k)\, D_{(k)}^{(km)}(e^{-z^k}) = \Gamma(1+1/k)e^{-z^k} H_{km}(z). \tag{2.1}$$

Therefore

$$\sum_{m=0}^{\infty} \frac{e^{-z^k} H_{km}(z)\, t^{mk}}{k^{2m}\Gamma(1+m)} = \frac{1}{\Gamma(1+1/k)} \int_0^\infty e^{-h^k(1+t^k)}{}_0F_1(1/k\,;\,-z^k h^k)\,dh$$

$$= \frac{e^{-(z^k/(1+t^k))}}{(1+t^k)^{1/k}} \quad \text{where } |t| < 1. \tag{2.2}$$

Now utilising (2.1) in (1.4), we may similarly derive:

$$e^{-z^k} H_{km+1}(z) = (-)^m \frac{k^{2(m+1)} z}{\Gamma(1+1/k)} \int_0^\infty e^{-h^k} h^{(m+1)k}{}_0F_1(1+1/k\,;\,-z^k h^k)\,dh$$

so that

$$\sum_{m=0}^{\infty} \frac{e^{-z^k} H_{km+1}(z)\, t^{mk}}{k^{2m}\Gamma(1+m)} = kz\, \frac{e^{-z^k/(1+t^k)}}{(1+t^k)^{1+1/k}} \quad (|t| < 1). \tag{2.3}$$

Finally we may put these results in the forms:

$$\sum_{m=0}^{\infty} \frac{H_{km}(z)\, t^{mk}}{k^{2m}\Gamma(1+m)} = \frac{e^{z^k t^k/(1+t^k)}}{(1+t^k)^{1/k}} \tag{2.4}$$

and

$$\sum_{m=0}^{\infty} \frac{H_{km+1}(z)t^{mk}}{k^{2m}\Gamma(1+m)} = kz\, \frac{e^{z^k t^k/(1+t^k)}}{(1+t^k)^{1+1/k}} \tag{2.5}$$

where $|t| < 1$, and consider them as the *generating functions* for $H_{km}(z)$ and $H_{km+1}(z)$ respectively.

3, Differential equations and Recurrence relations. By ordinary process of differentiation, we can easily prove that

$$D_{(k)}^{(m+1)k}(e^{-z^k}) + k\left\{zD_{(k)}^{(km+1)}(e^{-z^k}) + (km+1)D_{(k)}^{(km)}(e^{-z^k})\right\} = 0, \tag{3.1}$$

where the differential operators have their meanings as explained in § 1.

By (1.1), we therefore deduce·

$$D_{(k)}^{(k)}\{e^{-z^k}H_{km}(z)\} + k\left[z\frac{d}{dz}\{e^{-z^k}H_{km}(z)\} + (mk+1)e^{-z^k}H_{km}(z)\right] = 0.$$

Simplifying this relation, we finally see that $H_{km}(z)$ is a solution of the differential equation:

$$\frac{d^2w}{dz^2} + \left(\frac{2-k}{z} - kz^{k-1}\right)\frac{dw}{dz} + mk^2z^{k-2}w = 0. \tag{3.2}$$

Again from (1.4) we get

$$z^{k-2}H_{km+1}(z) - kz^{k-1}H_{km}(z) + H'_{km}(z) = 0$$

and if we differentiate this equation w. r. t. z and then utilise (3.2), we obtain·

$$H'_{km+1}(z) = k(km+1)H_{km}(z). \tag{3.3}$$

Differentiating (3.3) again w. r. t. z we finally see that $H_{km+1}(z)$ is a solution of the differential equation.

$$\frac{d^2w}{dz^2} - kz^{k-1}\frac{dw}{dz} + k(km+1)z^{k-2}w = 0. \tag{3.4}$$

Thus (3.2) and (3.4) are the differential equations satisfied by the generalised Hermite's polynomials $H_{km}(z)$ and $H_{km+1}(z)$ respectively. We may call these equations *Generalised Hermite's equations.*

Next to find the recurrence relations for these polynomials, we may at first note that (3.3) may readily be taken as one of such relations. As for others, we may proceed thus:

Simply combining (2.4) and (2.5), we get, after some easy simplifications, two other relations, *viz*:

$$H_{km+1}(z) - kz\, H_{km}(z) + mk^2\, H_{km-k+1}(z) = 0, \tag{3.5}$$

and
$$H'_{km}(z) = mk^2 z^{k-2} H_{km-k+1}(z). \tag{3.6}$$

Now differentiating (3.6) w. r. t. z and then eliminating $H''_{km}(z)$ by (3.2) we have

$$mk^2\{z^{k-2}H'_{km-k+1}(z) + (k-2)z^{k-3}H_{km-k+1}(z)\}$$

$$= \left\{kz^{k-1} + \frac{k-2}{z}\right\} H'_{km}(z) - mk^2 z^{k-2} H_{km}(z).$$

If we eliminate $H_{km-k+1}(z)$ and $H'_{km}(z)$ in this equation by (3.3) and (3.6) respectively, we finally get

$$H_{km}(z) - kz^{k-1}H_{km-k+1}(z) + k(km-k+1)H_{km-k}(z) = 0 \tag{3.7}$$

Thus (3.3) and (3.5)—(3.7) are the four recurrence relations connecting the generalised Hermite's polynomials $H_{km}(z)$ and $H_{km+1}(z)$.

4. Orthogonal properties. Multiplying both sides of the differential equation for $H_{km}(z)$ viz :

$$\frac{d^2w}{dz^2} + \left(\frac{2-k}{z} - kz^{k-1}\right)\frac{dw}{dz} + mk^2 z^{k-2} w = 0$$

by $e^{-z^k}z^{2-k}$, we may represent it in the form

$$\frac{d}{dz}\{z^{2-k} e^{-z^k} H'_{km}(z)\} + mk^2 e^{-z^k} H_{km}(z) = 0. \tag{4.1}$$

For $r \neq m$, we have similarly

$$\frac{d}{dz}\{z^{2-k} e^{-z^k} H'_{kr}(z)\} + rk^2 e^{-z^k} H_{kr}(z) = 0. \tag{4.2}$$

Now multiplying (4.1) by $H_{kr}(z)$ and (4.2) by $H_{km}(z)$ and then subtracting each other, we get finally

$$(m-r)k^2 e^{-z^k} H_{km}(z) H_{kr}(z) = \frac{d}{dz}\left[z^{2-k} e^{-z^k}\{H_{km}(z)H'_{kr}(z) - H_{kr}(z)H'_{km}(z)\}\right].$$

Integrating over $(0, \infty)$ we have therefore for $r \neq m$

$$\int_0^\infty e^{-z^k} H_{km}(z) H_{kr}(z)\, dz = 0. \tag{4.3}$$

Similarly multiplying (3.4) by e^{-z^k} and proceeding as before, we can prove that for $r \neq m$

$$\int_0^\infty e^{-z^k} z^{k-2} H_{km+1}(z) H_{kr+1}(z)\, dz = 0. \tag{4.4}$$

Now by (3.3), we have:

$$k(km+1)\int_0^\infty e^{-z^k}\{H_{km}(z)\}^2dz = \int_0^\infty e^{-z^k} H_{km}(z)H'_{km+1}(z)\,dz$$

$$= \int_0^\infty e^{-z^k}z^{k-2}\{H_{km+1}(z)\}^2\,dz, \tag{4.5}$$

—integrating by parts and then making use of (3.6) and (3.5) respectively.

Also by (3.6), we get

$$mk^2\int_0^\infty e^{-z^k}z^{k-2}\{H_{km-k-1}(z)\}^2dz = \int_0^\infty e^{-z^k}H_{km-k+1}(z)H'_{km}(z)dz$$

Integrating the R. S. by parts and utilising (3.3) and (3.7) we have similarly

$$\int_0^\infty e^{-z^k}\{H_{km}(z)\}^2dz = mk^2\int_0^\infty e^{-z^k}z^{k-2}\{H_{km-k+1}(z)\}^2dz. \tag{4.6}$$

Now changing m into $(m-1)$ in (4.5) and then combining the result with (4.6) we derive:

$$\int_0^\infty e^{-z^k}\{H_{km}(z)\}^2dz = mk^4(m+1/k-1)\int_0^\infty e^{-z^k}\{H_{k(m-1)}(z)\}dz,$$

from which it easily follows by induction that

$$\int_0^\infty e^{-z^k}\{H_{km}(z)\}^2dz = k^{4m-1}\Gamma(1+m)\Gamma(m+1/k). \tag{4.7}$$

Utilising (4.7) in (4.5) we further get

$$\int_0^\infty e^{-z^k}z^{k-2}\{H_{km+1}(z)\}^2dz = k^{4m+1}\Gamma(1+m)\Gamma(m+1/k+1). \tag{4.8}$$

Thus combining (4.3) with (4-7) and (4.4) with (4.8), we have finally

$$\left.\begin{aligned}\int_0^\infty e^{-z^k}H_{km}(z)H_{kr}(z)dz &= k^{4m-1}\Gamma(1+m)\,\Gamma(m+1/k).\\ &\qquad\qquad (\text{for } m=r)\\ &= 0\ (\text{otherwise}).\end{aligned}\right\} \tag{4.9}$$

and

$$\left.\begin{aligned}\int_0^\infty e^{-z^k}z^{k-2}H_{km+1}(z)H_{kr+1}(z)dz &= k^{4m+1}\Gamma(1+m)\Gamma(m+1/k+1)\\ &\qquad\qquad (\text{for } m=r)\\ &= 0\ (\text{otherwise}).\end{aligned}\right\} \tag{4.10}$$

From (4.9) and (4.10) we can now define two functions

$$\Phi_{km}(z) \equiv \frac{e^{-\frac{1}{2}z^2} H_{km}(z)}{\{k^{4m-1}\Gamma(1+m)\Gamma(m+1/k)\}^{\frac{1}{2}}}$$

and

$$\Phi_{km+1}(z) \equiv \frac{e^{-\frac{1}{2}z^2} z^{\frac{1}{2}k-1} H_{km+1}(z)}{\{k^{4m+1}\Gamma(1+m)\Gamma(m+1/k+1)\}^{\frac{1}{2}}}$$

which belong to the class $L^2(0, \infty)$ since $k > 0$, and form a *complete normal orthogonal system* for the interval $(0, \infty)$.

5. **Contour Integrals.** Consider the integral:

$$I \equiv \frac{1}{2\pi i} \int_{-\infty}^{(0+)} (z^k - t)^m e^t t^{-m-1/k} dt$$

where arg. z has its principal value and the path of integration starts at $-\infty$ on the real axis, encircles the origin once in the positive direction and returns to the initial point, so that the branch-point $t = z^k$ remains outside the contour.

Also $|\arg t| \leq \pi$, so that the integrand is made one-valued and that value of $\arg(z^k - t)$ is taken which $\to 0$ as $t \to \infty$, by a path lying inside the contour. The integrand evidently represents an analytic function of z under these conditions.

Now expanding $(z^k - t)^m$ and integrating term-by-term by the help of Hankel's formula:

$$\frac{1}{\Gamma(\nu + r + 1)} = \frac{1}{2\pi i} \int_{-\infty}^{(0+)} e^t t^{-(\nu+r+1)} \, dt$$

we easily prove that

$$I \equiv \frac{1}{\Gamma(m+1/k)} \left\{ z^{km} - \frac{km(km-k+1)}{1!\,k^2} z^{k(m-1)} + \frac{km(km-k+1)(km-k)(km-2k+1)}{2!\,k^4} \times z^{(m-2)} - \dots\dots \right\}$$

which by (1.7)

$$= \frac{H_{km}(z)}{k^{2m}\Gamma(m+1/k)}.$$

Thus we have:

$$H_{km}(z) = k^{2m} \frac{\Gamma(m+1/k)}{2\pi i} \int_{-\infty}^{(0+)} \left(\frac{z^k}{t} - 1 \right)^m e^t t^{-1/k} \, dt. \tag{5.1}$$

In like manner we can obtain:

$$H_{km+1}(z) = k^{2m+1} \frac{\Gamma(m+1/k+1)}{2\pi i} z \int_{-\infty}^{(0+)} \left(\frac{z^k}{t} - 1 \right)^m e^t t^{-1/k-1} \, dt. \tag{5.2}$$

The formula (5.1) and (5.2) may be looked upon as the analogues of Schläfli's contour integral for $J_\nu(z)$.

Next the integral

$$\int_0^1 e^{tz^k} t^{-m-1}(1-t)^{m+1/k-1}\,dt$$

where $m < 0$ and $m+1/k > 0$, may be put into the form:

$$\sum_{r=0}^{\infty} \frac{z^{kr}}{r!} \int_0^1 t^{r-m-1}(1-t)^{m+1/k-1}\,dt.$$

Evaluating by the help of Eulerian integral of the first kind, which is perfectly legitimate under the given conditions, we see that the above series is equal to

$$\sum_{r=0}^{\infty} \frac{z^{kr}}{r!} \frac{\Gamma(r-m)\Gamma(m+1/k)}{\Gamma(r+1/k)}.$$

Now making use of (1.6) and (1.7), we finally obtain:

$$H_{km}(z) = \frac{(-)^m k^{2m}}{\Gamma(-m)} \int_0^1 e^{tz^k} t^{-m-1}(1-t)^{m+1/k-1}\,dt \tag{5.3}$$

where $m < 0$ and $m+1/k > 0$. Similarly, we have

$$H_{km+1}(z) = \frac{(-)^m k^{2m+1}}{\Gamma(-m)} z \int_0^1 e^{tz^k} t^{-m-1}(1-t)^{m+1/k}\,dt. \tag{5.4}$$

where $m < 0$ and $m+1/k > -1$.

6. Next let us consider the integral

$$I \equiv \int_0^\infty z^{\nu+1} e^{-\frac{1}{2}z^2} J_\nu(az)\, H_{km}\left(\frac{z^{2/k}}{2^{1/k}}\right) dz$$

where $J_\nu(z)$ is the Bessel function of the first kind of order ν and $R(\nu) \geq -1$.

By (5.3) we have therefore

$$I \equiv \frac{(-)^m k^{2m}}{\Gamma(-m)} \int_0^\infty z^{\nu+1} e^{-\frac{1}{2}z^2} J_\nu(az)\,dz \int_0^1 t^{-m-1}(1-t)^{m+1/k-1} e^{\frac{1}{2}tz^2}\,dt$$

where $m < 0$, $m+1/k > 0$ and $R(\nu) > -1$.

Under these conditions the double integral is *absolutely convergent* and hence inverting the order of integration, which is quite permissible, we get

$$I \equiv \frac{(-)^m k^{2m}}{\Gamma(-m)} \int_0^1 t^{-m-1}(1-t)^{m+1/k-1}\,dt \int_0^\infty z^{\nu+1} e^{-\frac{1}{2}z^2(1-t)} J_\nu(az)\,dz$$

$$= \frac{(-)^m k^{2m} a^\nu}{\Gamma(-m)} \int_0^1 t^{-m-1}(1-t)^{m+1/k-\nu-2} e^{-\frac{1}{2}a^2/(1-t)}$$

Now changing the variable t into ζ according to the relation

$$t=\frac{\zeta}{1+\zeta}$$

we may rewrite the integral in the form:

$$I\equiv\frac{(-)^m k^{2m}a^{\nu}\,e^{-\frac{1}{4}a^2}}{\Gamma(-m)}\int_0^\infty \zeta^{-m-1}(1+\zeta)^{\nu+1-1/k}e^{-\frac{1}{4}\zeta a^2}\,d\zeta \tag{6.1}$$

where $m<0$, $m+1/k>0$ and $R(\nu)>-1$.

Also from the known integral (Whittaker and Watson, 1927)

$$W_{k,m}(z)=\frac{z^k e^{-\frac{1}{2}z}}{\Gamma(m+\frac{1}{2}-k)}\int_0^\infty t^{m-k-\frac{1}{2}}\left(1+\frac{t}{z}\right)^{m+k-\frac{1}{2}}e^{-t}\,dt,\quad R(m+\tfrac{1}{2}-k)>0$$

we easily deduce

$$\int_0^\infty \zeta^{-m-1}(1+\zeta)^{\nu+1-1/k}e^{-\frac{1}{4}a^2\zeta}\,d\zeta=2^{1+\frac{1}{2}\nu-\frac{1}{2}m-1/2k}\Gamma(-m)\,e^{\frac{1}{8}a^2}\,a^{m+1/k-\nu-2}$$
$$\times W_{1+\frac{1}{2}\nu+\frac{1}{2}m-1/2k,\ \frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}m-1/2k}(\tfrac{1}{4}a^2)$$

so that (6.1) can be easily thrown into the form:

$$\int_0^\infty z^{\nu+1}e^{-\frac{1}{2}z^2}J_\nu(az)\;H_{km}\left(\frac{z^{2/k}}{2^{1/k}}\right)dz=(-)^m k^{2m}2^{1+\frac{1}{2}(\nu-m-1/k)}\,e^{-\frac{1}{8}a^2}$$
$$\times a^{m+1/k-2}W_{1+\frac{1}{2}\nu+\frac{1}{2}m-1/2k,\ \frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}m-1/2k}(\tfrac{1}{4}a^2) \tag{6.2}$$

where $m<0$, $m+1/k>0$ and $R(\nu)>-1$.

Now by applying the principle of analytic continuation, we may see that the integral (6.2) can hold only when $1/k>0$ and $R(\nu)>-1$.

Similarly starting with the integral

$$\int_0^\infty z^{\nu+1-2/k}e^{-\frac{1}{2}z^2}J_\nu(az)H_{km+1}\left(\frac{z^{2/k}}{2^{1/k}}\right)dz$$

and proceeding as before, we can prove that

$$\int_0^\infty z^{\nu+1-2/k}e^{-\frac{1}{2}z^2}J_\nu(az)H_{km+1}\left(\frac{z^{2/k}}{2^{1/k}}\right)dz=(-)^m k^{2m+1}2^{\frac{1}{2}(\nu-m+1-3/k)}$$
$$e^{-\frac{1}{8}a^2}a^{m+1/k-1}W_{\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}m-1/2k,\ \frac{1}{2}\nu-\frac{1}{2}m-1/2k}(\tfrac{1}{4}a^2) \tag{6.3}$$

where $1/k>-1$ and $R(\nu)>-1$.

The integrals (6.2) and (6.3) can obviously be represented in more elegant forms as:

$$\int_0^\infty z^{\frac{1}{2}k\nu+k-1}e^{-z^2}J_\nu(az^{k/2})\,H_{km}(z)\,dz=(-)^m k^{2m-1}2^{2-m-1/k}a^{m+1/k-2}$$
$$e^{-\frac{1}{8}a^2}W_{1+\frac{1}{2}\nu+\frac{1}{2}m-1/2k,\ \frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}m-1/2k}(\tfrac{1}{4}a^2) \tag{6.4}$$

where $1/k > 0$ and $R(\nu) > -1$; and

$$\int_0^\infty z^{\frac{1}{2}k\nu+k-2} J_\nu(az^{k/2}) H_{km+1}(z)\, dz = (-)^m k^{2m} 2^{1-m-1/k} a^{m+1/k-1}$$

$$e^{-\frac{1}{4}a^2} W_{\frac{1}{4}+\frac{1}{2}\nu+\frac{1}{2}m-1/2k,\ \frac{1}{2}\nu-\frac{1}{2}m-1/2k}(\tfrac{1}{4}a^2) \qquad (6{\cdot}5)$$

where $1/k > -1$ and $R(\nu) > -1$.

Thus we evaluate two infinite integrals (6·4) and (6·5) involving the product of generalised Hermite's polynomials with Bessel function of the first kind of order ν in terms of Whittaker's function $W_{k,m}(z)$.

Now replacing the $W_{k,m}$ – functions on the R. S. of (6.2) and (6.3) by ${}_1F_1$ – functions according to the formula:

$$z^{\frac{1}{2}\alpha} {}_1F_1(-m\,;\,\alpha\,;\,z) = \frac{\Gamma(1-m-\alpha)}{\Gamma(1-\alpha)} e^{\frac{1}{2}z} W_{m+\frac{1}{2}\alpha,\ \frac{1}{2}(1-\alpha)}(z)$$

and then making use of (1.8) and (1.5), we easily see that

$$z^{\nu+\frac{1}{2}} e^{-\frac{1}{2}z^2} H_{k(\nu+2)-2}\left(\frac{z^{2/k}}{2^{1/k}}\right)$$

and

$$z^{\nu+\frac{1}{2}-2/k} e^{-\frac{1}{2}z^2} H_{k\nu-1}\left(\frac{z^{2/k}}{2^{1/k}}\right)$$

are $\pm R_\nu$ *i.e., self-reciprocal or skew self-reciprocal* according as ν is *even* or *odd*.

7. We next evaluate some infinite integrals involving the product of several *generalised* Hermite's polynomials of the types:

$$I^{a_1,\ldots\ldots,a_r;\,p}_{k(m_1,\ldots\ldots,m_r)} \equiv \int_0^\infty e^{-pz^2} H_{km_1}(a_1 z) H_{km_2}(a_2 z) \ldots H_{km_r}(a_r z)\, dz \qquad (7.1)$$

and

$$I^{a_1,\ldots\ldots,a_r;\,p,\,l.}_{k(m_1+1,\ldots\ldots,m_r+1)} \equiv \int_0^\infty e^{-pz^2} z^{l-1} H_{km_1+1}(a_1 z) H_{km_2+1}(a_2 z) \ldots$$

$$\ldots H_{km_r+1}(a_r z)\, dz \qquad (7.2)$$

where p, l and a_i (for $i = 1, 2, \ldots, r$) are all *positive*.

We consider *three* cases.

Case I. When $r = 1$, (7.1) and (7.2) assume the forms:

$$I^{a;\,p}_{km} \equiv \int_0^\infty e^{-pz^2} H_{km}(az)\, dz$$

and

$$I^{a;\,p,\,1}_{km+1} \equiv \int_0^\infty e^{-pz^2} z^{k-1} H_{km+1}(az)\, dz.$$

Expanding $H_{km}(az)$ by (1.7) and then integrating term-by-term, we get

$$I_{km}^{a\,;\,p} = k^{2m-1}\frac{\Gamma(m+1/k)(a^k-p)^m}{p^{m+1/k}}$$

Similarly

$$I_{km+1}^{a\,;\,p,\,1} = ak^{2m}\frac{\Gamma(m+1/k+1)(a^k-p)^m}{p^{m+1/k+1}}$$

We have therefore the following *operational representations*;

$$z^{1/k-1}H_{km}(z^{1/k}) \doteqdot k^{2m}\Gamma(m+1/k)\,\frac{(1-p)^m}{p^{m+1/k}}$$

and

$$H_{km+1}(z^{1/k}) \doteqdot k^{2m+1}\Gamma(m+1/k+1)\,\frac{(1-p)^m}{p^{m+1/k+1}}$$

Case II. When $r = 2$, (7.1) and (7.2) become

$$I_{k(m_1,\,m_2)}^{a_1,\,a_2\,;\,p.} \equiv \int_0^\infty e^{-pz^k} H_{km_1}(a_1z)H_{km_2}(a_2z)\,dz.$$

and

$$I_{k(m_1+1,\,m_2+1)}^{a_1,\,a_2\,;\,p,\,2} \equiv \int_0^\infty e^{-pz^k} z^{k-2} H_{km_1+1}(a_1z)H_{km_2+1}(a_2z)\,dz.$$

Expanding $H_{km_1}(a_1z)$ and $H_{km_2}(a_2z)$ by (1.7) and then making use of (1.6), we have

$$I_{k(m_1,\,m_2)}^{a_1,\,a_2\,;\,p} = k^{2(m_1+m_2)}a_1^{km_1}a_2^{km_2}\sum_{r=0}^{m_1}\sum_{s=0}^{m_2}\frac{(-)^{r+s}(-m_1)_r(1-1/k-m_1)_r(-m_2)_s(1-1/k-m_2)_s}{r!\;s!\;a_1^{kr}\,a_2^{ks}}\times$$

$$\int_0^\infty e^{-pz^k} z^{k(m_1+m_2-r-s)}dz.$$

$$= k^{2(m_1+m_2)-1}a_1^{km_1}a_2^{km_2}\times$$

$$\frac{\Gamma(m_1+m_2+1/k)}{p^{m_1+m_2+1/k}}\sum_{r=0}^{m_1}\sum_{s=0}^{m_2}\frac{(-m_1)_r(1-1/k-m_1)_r(-m_2)_s(1-1/k-m_2)_s}{r!\;s!\;(1-m_1-m_2-1/k)_{r+s}}\,(p/a_1^k)^r(p/a_2^k)^s$$

$$= k^{2(m_1+m_2)-1}a_1^{km_1}a_2^{km_2}\times$$

$$\frac{\Gamma(m_1+m_2+1/k)}{p^{m_1+m_2+1/k}}\;F_3[-m_2,\,1-1/k-m_1\,;\,1-1/k-m_2,\;-m_1\,;\,1-m_1-m_2-1/k\,;$$

$$p/a_2^k,\,p/a_1^k]$$

where F_3 is Appell's hypergeometric function of two variables. Using the relation (Bailey, 1935)

$$F_3[\alpha,\,\gamma-\alpha\,;\,\beta,\,\gamma-\beta\,;\,\gamma\,;\,x,\,y] = (1-y)^{\alpha+\beta-\gamma}F[\alpha,\,\beta\,;\,\gamma\,;\,x+y-xy]$$

we have

$$I_{k(m_1, m_2)}^{a_1, a_2; p} = k^{2(m_1+m_2)-1} \frac{\Gamma(m_1+m_2+1/k)(a_1a_2)^{km_2}(a_1{}^k-p)^{m_1-m_2}}{p^{m_1+m_2+1/k}} \times$$

$$F\left[-m_2, 1-1/k-m_2; 1-m_1-m_2-1/k; p\frac{a_1{}^k+a_2{}^k-p}{(a_1a_2)^k}\right] \qquad (7.3)$$

Similarly from (1.8)

$$I_{k(m_1+1, m_2+1)}^{a_1, a_2; p, 2} = k^{2(m_1+m_2)+1} \frac{\Gamma(m_1+m_2+1/k+1)(a_1{}^k-p)^{m_1-m_2}}{p^{m_1+m_2+1/k+1}}(a_1a_2)^{km_2+1} \times$$

$$F\left[-m_2-1/k, -m_2; -m_1-m_2-1/k; p\frac{a_1{}^k+a_2{}^k-p}{(a_1a_2)^k}\right] \qquad (7.4)$$

Now, if in particular, we put $k=2$ and $p=1$ in (7.3) and (7.4), we get Bailey's (1948) result (1.3) which also includes those of Busbridge (1948) and Titchmarsh (1948).

Case III. Now it is clear that by the method as in Case II, the general integrals (7.1) and (7.2) can be evaluated as a Lauricella's hypergeometric function of several variables for general values of the a's. But we next propose to evaluate two more general integrals in a different method.

From (1.1) we at first notice that

$$\int_0^\infty e^{-z^k} H_{km}(z) f(z)\, dz = \int_0^\infty D_{(k)}^{k(m-1)} (e^{-z^k}) D_{(k)}^{(k)}\{f(z)\} dz$$

where $f(z)$ is a polynomial. Proceeding in this manner, we have finally

$$\int_0^\infty e^{-z^k} H_{km}(z) f(z)\, dz = \int_0^\infty e^{-z^k} D_{(k)}^{(km)} \{f(z)\} dz \qquad (7.5)$$

where $f(z)$ is a polynomial.

If we now suppose that

$$f(z) = H_{km_1}(a_1z) H_{km_2}(a_2z) \ldots H_{km_r}(a_rz) \quad (r > 2)$$

we can easily verify from (1.7) that

$$D_{(k)}^{(km)}\{f(z)\} = 0 \text{ if } m > \sum_{i=1}^{r} m_i$$

$$= k^{4m} \frac{\Gamma(m+1)\Gamma(m+1/k)}{\Gamma(1/k)} a_1{}^{km_1} a_2{}^{km_2} \ldots a_r{}^{km_r} \text{ if } m = \sum_{i=1}^{r} m_i.$$

But when $m < \sum_{i=1}^{r} m_i$ we have complications depending on the difference $(m - \Sigma m_i)$ and we do not propose to discuss this case here.

From (7.5) we therefore get

$$\left.\begin{aligned} I^{1,a_1,\ldots,a_r;1}_{k(m,m_1,\ldots,m_r)} &= 0 \quad \text{if } m > \Sigma m_i \\ &= k^{4m-1} a_1^{km_1} a_2^{km_2} \ldots a_r^{km_r} \Gamma(m+1)\Gamma(m+1/k) \\ &\text{if } m = \Sigma m_i (r > 2) \end{aligned}\right\} \tag{7.6}$$

Similarly, we have

$$\left.\begin{aligned} I^{1,a_1,\ldots,a_r;1,2}_{k(m+1,m_1+1,\ldots,m_r+1)} &= 0 \text{ if } km+1 = k\sum_{i=1}^{r} m_i + r \\ &= k^{4m+2/k+(1-2/k)r} \times \\ a_1^{km_1+1} a_2^{km_2+1} \ldots a_r^{km_r+1} & \frac{\Gamma(m+1/k+1)\Gamma(m+2/k)}{\Gamma(2/k)} \end{aligned}\right\} \tag{7.7}$$

$$\text{if } km+1 = k\Sigma m_i + r.$$

Now putting $k=2$ in (7.6) and (7.7) we deduce the result due to Lord (1949).

8. Lastly we have

$$\int_0^\infty e^{-z^k} H_{km}(z)\, {}_pF_q(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;tz^k)\,dz$$

($p \leq q$ and $|t| < 1$)

$$= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_p)_n\, t^n}{(\beta_1)_n \ldots (\beta_q)_n\, n!} \int_0^\infty e^{-z^k} H_{km}(z)\, z^{kn}\,dz \tag{8.1}$$

(term-by-term integration being valid under the specified conditions.)

Now by (7.5)

$$\int_0^\infty e^{-z^k} H_{km}(z) z^{kn}\,dz = \int_0^\infty e^{-z^k} D^{(km)}_{(k)}(z^{kn})\,dz$$

and actually operating by $D^{(k)}_{(k)}$ on z^{kn} m times successively, we see that

$$D^{(km)}_{(k)}(z^{kn}) = k^{2m} \frac{\Gamma(n+1)\Gamma(n+1/k)\, z^{k(n-m)}}{\Gamma(n-m+1/k)\Gamma(n-m+1)}$$

for $n \geq m$. Therefore from (8.1), we get finally

$$\int_0^\infty e^{-z^k} H_{km}(z)\, {}_pF_q(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;tz^k)\,dz$$

($p \leq q$ and $|t| < 1$)

$$= t^m k^{2m-1}\Gamma(m+1/k) \frac{(\alpha_1)_m \ldots (\alpha_p)_m}{(\beta_1)_m \ldots (\beta_q)_m} {}_{p+1}F_q(\alpha_1+m,\ldots,\alpha_p+m,\, m+1/k;$$

$$\beta_1+m,\ldots,\beta_q+m;t) \tag{8.2}$$

Similarly, we have

$$\int_0^\infty e^{-z^2} z^{k-1} H_{km+1}(z)\ {}_pF_q(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots\beta_q;tz^k)\,dz$$

$$=t^m k^{2m}\frac{\Gamma(m+1/k+1)\Gamma(m+2/k)}{\Gamma(m+1)\Gamma(2/k)}\frac{(\alpha_1)_m\ldots(\alpha_p)_m}{(\beta_1)_m\ldots(\beta_q)_m}\times$$

$$ {}_{p+3}F_{q+2}(\alpha_1+m,\ldots,\alpha_p+m,\ m+1/k+1,\ m+2/k,1;\ \beta_1+m,\ldots,\beta_q+m,\ m+1,\ 2/k;t) \tag{8.3}$$

where $p \leq q$ and $|t|<1$.

We shall next consider some particular cases of (8.2) and (8.3).

* Since for all values of μ and ν

$$J_\mu(z)\,J_\nu(z)=\sum_{m=0}^{\infty}\frac{(-)^m\Gamma(\mu+\nu+2m+1)(\tfrac{1}{2}z)^{\mu+\nu+2m}}{m!\,\Gamma(\mu+m+1)\Gamma(\nu+m+1)\Gamma(\mu+\nu+m+1)}$$

we have $\Gamma(1+\mu)\Gamma(1+\nu)\left(\frac{2}{z}\right)^{\mu+\nu}J_\mu(z)J_\nu(z)$

$$= {}_2F_3\left(\frac{1+\mu+\nu}{2},\ \frac{2+\mu+\nu}{2};1+\mu,1+\nu,1+\mu+\nu;-z^2\right)$$

and therefore we deduce from (8.2) and (8.3) the following integrals:

$$\int_0^\infty e^{-z^2}H_{km}(z)J_\mu[(\tfrac{1}{2}z^k)^{\frac{1}{2}}]J_\nu[(\tfrac{1}{2}z^k)^{\frac{1}{2}}]z^{-\frac{1}{2}k(\mu+\nu)}\,dz$$

$$=\frac{(-)^m k^{2m-1}\Gamma(m+1/k)\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\mu+\tfrac{1}{2}\nu+m)\Gamma(1+\tfrac{1}{2}\mu+\tfrac{1}{2}\nu+m)}{2^{m+\frac{1}{2}(\mu+\nu)}\,\Gamma(\tfrac{1}{2})\Gamma(1+\mu+m)\Gamma(1+\nu+m)\Gamma(1+\mu+\nu+m)}\times$$

$$ {}_3F_3(\tfrac{1}{2}+\tfrac{1}{2}\mu+\tfrac{1}{2}\nu+m,\ 1+\tfrac{1}{2}\mu+\tfrac{1}{2}\nu+m,\ m+1/k;1+\mu+m,\ 1+\nu+m,1+\mu+\nu+m;-\tfrac{1}{2}) \tag{8.4}$$

and $$\int_0^\infty e^{-z^2}H_{km+1}(z)J_\mu[(\tfrac{1}{2}z^k)^{\frac{1}{2}}]J_\nu[(\tfrac{1}{2}z^k)^{\frac{1}{2}}]z^{k(1-\frac{1}{2}\mu-\frac{1}{2}\nu)-1}\,dz$$

$$=\frac{(-)^m k^{2m}\Gamma(m+1/k+1)\Gamma(m+2/k)\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\mu+\tfrac{1}{2}\nu+m)\Gamma(1+\tfrac{1}{2}\mu+\tfrac{1}{2}\nu+m)}{2^{m+\frac{1}{2}(\mu+\nu)}\Gamma(m+1)\Gamma(2/k)\Gamma(\tfrac{1}{2})\Gamma(1+\mu+m)\Gamma(1+\nu+m)\Gamma(1+\mu+\nu+m)}\times$$

$$ {}_5F_5(\tfrac{1}{2}+\tfrac{1}{2}\mu+\tfrac{1}{2}\nu+m,\ 1+\tfrac{1}{2}\mu+\tfrac{1}{2}\nu+m,\ m+1/k+1,\ m+2/k,\ 1;\ 1+\mu+m,\ 1+\nu+m,1+\mu+\nu+m,\ 1+m,\ 2/k;-\tfrac{1}{2}) \tag{8.5}$$

Next since (Meijer, 1935)

$$ {}_0F_3(\tfrac{1}{2}+\mu,1+\mu,\ 2\mu+1;-z^2/6^4)\ \frac{z^{2\mu}}{\{\Gamma(1-2\mu)\}^2\,2^{4\mu}}=I_{2\mu}(\sqrt{z})\,J_{2\mu}(\sqrt{z})$$

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* Whittaker and Watson, loc. cit p. 380. Ex. 9.

we have similarly from (8.2) and (8.3)

$$\int_0^\infty e^{-z^2} H_{km}(z) I_\mu(z^{\frac{1}{2}k}) J_\mu(z^{\frac{1}{2}k}) z^{-\frac{1}{2}k\mu}\, dz$$

$$= \frac{(-)^m k^{2m-1}\Gamma(m+1/k)\Gamma(\frac{1}{2})\{\Gamma(1+\mu)\}^2}{2^{3\mu+4m}3^{4m}\Gamma(\frac{1}{2}+\frac{1}{2}\mu+m)\Gamma(1+\frac{1}{2}\mu+m)\{\Gamma(1-\mu)\}^2\Gamma(1+\mu+m)} \times$$

$$ {}_1F_3(m+1/k\,;\, \tfrac{1}{2}+\tfrac{1}{2}\mu+m,\ 1+\tfrac{1}{2}\mu+m,\ 1+\mu+m\,;\, -1/6^4) \qquad (8.6)$$

and $$\int_0^\infty e^{-z^2} H_{km+1}(z) I_\mu(z^{\frac{1}{2}k}) J_\mu(z^{\frac{1}{2}k}) z^{k(1-\frac{1}{2}\mu)-1}\, dz$$

$$= \frac{(-)^m k^{2m}\Gamma(m+1/k+1)\Gamma(m+2/k)\Gamma(\frac{1}{2})\{\Gamma(1+\mu)\}^2\{\Gamma(1-\mu)\}^{-2}}{2^{3\mu+4m}3^{4m}\Gamma(m+1)\Gamma(2/k)\Gamma(\frac{1}{2}+\frac{1}{2}\mu+m)\Gamma(1+\frac{1}{2}\mu+m)\Gamma(1+\mu+m)} \times$$

$$ {}_3F_5(m+1/k+1,\ m+2/k,\ 1\,;\, \tfrac{1}{2}+\tfrac{1}{2}\mu+m,\ 1+\tfrac{1}{2}\mu+m,$$
$$1+\mu+m,\ m+1,\ 2/k\,;\, -1/6^4) \qquad (8.7)$$

Again since *

$$ {}_2F_3(\alpha, \varrho-\alpha\,;\, \varrho, \tfrac{1}{2}\varrho, \tfrac{1}{2}\varrho+\tfrac{1}{2}, \tfrac{1}{4}z^2)$$
$$= e^{-z}{}_1F_1(\alpha\,;\, \varrho\,;\, z)\ {}_1F_1(\varrho-\alpha\,;\, \varrho\,;\, z)$$

we get, after a slight change of variable, from (8.2) and (8.3)

$$\int_0^\infty e^{-z^2-z} H_{km}(z^{2/k})\ {}_1F_1(\alpha\,;\, \varrho\,;\, z)\ {}_1F_1(\varrho-\alpha\,;\, \varrho\,;\, z)\, z^{2/k-1}\, dz$$

$$= \frac{\pi^{\frac{1}{2}}k^{2m}\{\Gamma(\varrho)\}^2\Gamma(\alpha+m)\Gamma(\varrho-\alpha+m)\Gamma(m+1/k)}{2^{2m+\varrho}\Gamma(\alpha)\Gamma(\varrho-\alpha)\Gamma(\varrho+m)\Gamma(\frac{1}{2}\varrho+m)\Gamma(\frac{1}{2}\varrho+\frac{1}{2}+m)} \times$$

$$ {}_3F_3\{\alpha+m,\ \varrho-\alpha+m,\ m+1/k\,;\, \varrho+m,\ \tfrac{1}{2}\varrho+m,\ \tfrac{1}{2}\varrho+\tfrac{1}{2}+m\,;\, \tfrac{1}{4}\} \qquad (8.8)$$

and $$\int_0^\infty e^{-z^2-z} H_{km+1}(z^{2/k})\ {}_1F_1(\alpha\,;\, \varrho\,;\, z)\ {}_1F_1(\varrho-\alpha\,;\, \varrho\,;\, z) z\, dz$$

$$= \frac{\pi^{\frac{1}{2}}k^{2m+1}\{\Gamma(\varrho)\}^2\Gamma(\alpha+m)\Gamma(\varrho-\alpha+m)\Gamma(m+1/k+1)\Gamma(m+2/k)}{2^{2m+\varrho}\Gamma(\alpha)\Gamma(\varrho-\alpha)\Gamma(\varrho+m)\Gamma(\frac{1}{2}\varrho+m)\Gamma(\frac{1}{2}\varrho+\frac{1}{2}+m)\Gamma(m+1)\Gamma(2/k)} \times$$

$$ {}_5F_5(\alpha+m,\ \varrho-\alpha+m,\ m+1/k+1,\ m+2/k,\ 1\,;\, \varrho+m,\ \tfrac{1}{2}\varrho+m,$$
$$\tfrac{1}{2}\varrho+\tfrac{1}{2}+m,\ m+1,\ 2/k\,;\, \tfrac{1}{4}) \qquad (8.9)$$

Now as $$L_n^{(\alpha)}(z) = \frac{\Gamma(\alpha+1+n)}{n!\,\Gamma(\alpha+1)}\ {}_1F_1(-n\,,\, \alpha+1\,;\, z)$$

$$M_{k,m}(z) = z^{m+\frac{1}{2}}e^{-\frac{1}{2}z}\ {}_1F_1(m+1/2-k\,;\, 2m+1\,;\, z)$$

and † $$E(\alpha, \beta :: z) = \Gamma(\alpha)\Gamma(\beta-\alpha)\, z^\alpha\ {}_1F_1(\alpha\,;\, 1+\alpha-\beta\,;\, z)$$

* Bailey (1935) loc. cit. p 97. Ex. 4. (i)

† It is the E function of MacRobert (1942)

we can easily deduce from (8.8) and (8.9) the following special cases:

$$(i)\quad \int_0^\infty e^{-z^2-\frac{1}{2}z} z^{2/k-(\alpha+3)/2} H_{km}(z^{2/k}) L_n^{(\alpha)}(z) M_{-n-\frac{1}{2}(\alpha+1),\,\frac{1}{2}\alpha}(z)\,dz$$

$$=\frac{(-)^m \pi^{\frac{1}{2}} k^{2m}\Gamma(m+1/k)\Gamma(\alpha+1)\Gamma(\alpha+1+m+n)}{2^{2m+\alpha+1}\Gamma(\alpha+m+1)\Gamma(m+\frac{1}{2}\alpha+\frac{1}{2})\Gamma(m+1+\frac{1}{2}\alpha)\Gamma(n-m+1)}\times$$

$$\times\,{}_3F_3(m-n,\ m+n+\alpha+1,\ m+1/k\,;\ m+\alpha+1,\ m+\tfrac{1}{2}\alpha+\tfrac{1}{2},\ m+1+\tfrac{1}{2}\alpha\,;\ \tfrac{1}{4})$$

$$(ii)\quad \int_0^\infty e^{-z^2-\frac{1}{2}z} z^{\frac{1}{2}(1-\alpha)} H_{km+1}(z^{2/k}) L_n^{(\alpha)}(z) M_{-n-\frac{1}{2}(\alpha+1),\,\frac{1}{2}\alpha}(z)\,dz$$

$$=\frac{(-)^m \pi^{\frac{1}{2}} k^{2m+1}\Gamma(m+1/k+1)\Gamma(m+2/k)\Gamma(\alpha+1)\Gamma(\alpha+1+m+n)}{2^{2m+\alpha+1}\Gamma(m+1)\Gamma(2/k)\Gamma(\alpha+1+m)\Gamma(\frac{1}{2}\alpha+\frac{1}{2}+m)\Gamma(\frac{1}{2}\alpha+1+m)\Gamma(n-m+1)}$$

$${}_5F_5(m-n,\ \alpha+1+m+n,\ m+1/k+1,\ m+2/k,\ 1\,;$$

$$\alpha+1+m,\ \tfrac{1}{2}\alpha+\tfrac{1}{2}+m,\ \tfrac{1}{2}\alpha+1+m,\ m+1,\ 2/k,\ \tfrac{1}{4}).$$

$$(iii)\quad \int_0^\infty e^{-z^2-z/2} z^{2/k-1-\frac{1}{2}(3\alpha+\beta)} H_{km}(z^{2/k}) E(\alpha,\ 1-\beta :: z)\, W_{\frac{1}{2}(\alpha-\beta),\,\frac{1}{2}(1-\alpha-\beta)}(z)\,dz$$

$$=\frac{\pi^{\frac{1}{2}} k^{2m}\Gamma(m+1/k)\Gamma(\alpha+m)\Gamma(\beta+m)\{\Gamma(\alpha+\beta)\Gamma(1-\alpha-\beta)\}^2}{2^{2m+\alpha+\beta}\Gamma(1-\alpha)\Gamma(\beta)\Gamma(\alpha+\beta+m)\Gamma(\frac{1}{2}\alpha+\frac{1}{2}\beta+m)\Gamma(\frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{1}{2}+m)}\times$$

$${}_3F_3(\alpha+m,\ \beta+m,\ m+1/k\,;\ \alpha+\beta+m,\ \tfrac{1}{2}\alpha+\tfrac{1}{2}\beta+m,\ \tfrac{1}{2}\alpha+\tfrac{1}{2}\beta+\tfrac{1}{2}+m,\ \tfrac{1}{4})$$

$$(iv)\quad \int_0^\infty e^{-z^2-z/2} z^{1-(3\alpha+\beta)/2} H_{km+1}(z^{2/k}) E(\alpha,\ 1-\beta :: z)\, W_{\frac{1}{2}(\alpha-\beta),\,\frac{1}{2}(1-\alpha-\beta)}(z)\,dz$$

$$=\frac{\pi^{\frac{1}{2}} k^{2m+1}\Gamma(m+1/k+1)\Gamma(m+2/k)\Gamma(\alpha+m)\Gamma(\beta+m)\{\Gamma(\alpha+\beta)\Gamma(1-\alpha-\beta)\}^2}{2^{2m+\alpha+\beta}\Gamma(1-\alpha)\Gamma(m+1)\Gamma(2/k)\Gamma(\beta)\Gamma(\alpha+\beta+m)\Gamma(\frac{1}{2}\alpha+\frac{1}{2}\beta+m)\Gamma(\frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{1}{2}+m)}$$

$$\times\,{}_5F_5(\alpha+m,\ \beta+m,\ m+1/k+1,\ m+2/k,\ 1\,;\ \alpha+\beta+m,\ \tfrac{1}{2}\alpha+\tfrac{1}{2}\beta+m,$$

$$\tfrac{1}{2}\alpha+\tfrac{1}{2}\beta+\tfrac{1}{2}+m,\ m+1,\ 2/k\,;\ \tfrac{1}{4}).$$

In the same way, by replacing the ${}_1F_1$-functions in (8.8) and (8.9) by those special functions which are expressible in terms of ${}_1F_1$, *e g.*, Sonine's polynomial, Bateman's functions, Weber's parabolic Cylinder functions, etc. we can obtain many other interesting integrals.

In conclusion, I wish to express my grateful thanks to Dr. S. C. Mitra for his helpful criticisms and advice.

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References.

Bailey, W. N (1935), Generalised Hypergeometric Series. No 32. 80 (5).
——— (1948), *Jour. Lond. Math. Soc.*, **23**, 292
Busbridge, I. W. (1948), *Jour. Lond Math. Soc*, **23**, 136, (2.2).
Lord, R. D (1949), *Jour. Lond. Math. Soc.*, **24**, (7)
MacRobert, T. M., (1942). *Quart. Jour Math.*, **13**, 65.
Meijer, C. S. (1935), *Proc. Lond. Math Soc.*, **40**, 11.
Titchmarsh, E. C. (1948), *Jour Lond. Math. Soc.*, **23**, 15—16.
Whittaker and Watson (1927), Modern Analysis, 340.

CERTAIN THEOREMS ON SELF-RECIPROCAL FUNCTIONS

BY

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(*Communicated by Dr S. C. Mitra, Received—May 19, 1953*)

1. The object of this paper is to give certain theorems on self-reciprocal functions under the new transform and the generalised transform with the kernels $\varpi_{\mu,\nu}(x)$ and $\varpi_{n_1, n_2 \ldots n_n}(x)$ defined as

(i) $$\varpi_{\mu,\nu}(xy) = (xy)^{\frac{1}{2}} \int_0^\infty J_\nu(t)\, J_\mu\left(\frac{xy}{t}\right)\frac{dt}{t} \qquad (\mu+\tfrac{1}{2} > 0,\ \nu+\tfrac{1}{2} > 0)$$

(ii) $$\varpi_{n_1, n_2, \ldots n_n}(xy) = \int_0^\infty \frac{J^{n}_{n}(t)}{t^{\frac{1}{2}}}\, \varpi_{n_1, n_2 \ldots n_{n-1}}\left(\frac{xy}{t}\right) dt \quad (n_n+\tfrac{1}{2} > 0;\ n=1, 2,\ldots)$$

In a recent paper [Bhatnagar, 1954 (1)] we have proved in detail that the kernal $\varpi_{\mu,\nu}(x)$ does play the role of a transform. In another paper [Bhatnagar, 1953 (2)] we have hinted that the generalised kernal $\varpi_{n_1, n_2 \ldots n_n}(xy)$ might also play role of a transform but we have not shown that it plays the role of a transform. Here we shall give a formal proof to show that $\varpi_{n_1, n_2 \ldots n_n}(x)$ does play the role of a transform. The detailed proof can also be given on the lines by Titchmarsh (1948) or by us [Bhatnagar, 1954 (1)].

2. We shall now show that the integral

$$\varpi_{n_1 n_2 \ldots n_n}(xz) = (xz)^{\frac{1}{2}} \int_0^\infty \int_0^\infty \ldots \int_0^\infty J_{n_1}(t_1)\, J_{n_2}(t_2) \ldots J_{n_n-1}(t_{n-1})\, J_{n_n}\left(\frac{xz}{t_1 t_2 \ldots t_{n-1}}\right) \frac{dt_1 dt_2 \ldots dt_{n-1}}{t_1 t_2 \ldots t_{n-1}}$$

can be regarded as a kernel which gives rise to a transform, at any rate formally. This transform has not, I think, been noticed previously.

$$\text{Let,} \quad R(n_1+\tfrac{1}{2}) > 0,\ R(n_2+\tfrac{1}{2}) > 0, \ldots\ldots R(n_n+\tfrac{1}{2}) > 0.$$

and define $\varpi_{n_1, n_2 \ldots n_n}(xy)$ by the above integral

$$g(x) = \int_0^\infty \varpi_{n_1 n_2 \ldots n_n}(xz) f(z)\, dz.$$

Therefore,

$$\int_0^\infty \varpi_{n_1 n_2 \ldots n_n}(xz) g(x)\, dz = \int_0^\infty \int_0^\infty \varpi_{n_1 n_2 \ldots n_n}(xz)\, \varpi_{n_1 n_2 \ldots n_n}(zp) f(p)\, dp\, dz$$

$$I = \int_0^\infty \int_0^\infty \ldots \int_0^\infty (xz)^{\frac{1}{2}} (zp)^{\frac{1}{2}}\, J_{n_1}(t_1)\, J_{n_2}(t_2) \ldots J_{n_n-1}(t_{n-1})\, J_{n_n}\left(\frac{xt}{t_1\, t_2 \ldots t_{n-1}}\right)$$

$$\times\ J_{n_1}(T_1)\, J_{n_2}(T_2) \ldots J_{n_n-1}(T_{n-1})\, J_{n_n}\left(\frac{zp}{T_1 T_2 \ldots T_{n-1}}\right) f(p)\, \frac{dp\, dz\, dt_1 dt_2 \ldots dt_{n-1}}{t_1\, t_2\, t_3 \ldots t_{n-1}}\, \frac{dT_1\, dT_2 \ldots dT_{n-1}}{T_1\, T_2 \ldots T_{n-1}}$$

Now

$$\int_0^\infty\int_0^\infty\left(\frac{pz}{T_1T_2\ldots T_{n-1}}\right)^{\frac{1}{2}}J_{n_n}\left(\frac{pz}{T_1T_2\ldots T_{n-1}}\right)\left(\frac{xz}{t_1t_2\ldots t_{n-1}}\right)^{\frac{1}{2}}J_{n_n}\left(\frac{xz}{t_1\,t_2\ldots t_{n-1}}\right)f(p)\,dp\,dz$$

$$=T_1T_2\ldots T_{n-1}\int_0^\infty\int_0^\infty J_{n_n}\left(\frac{xzT_1\,T_2\ldots T_{n-1}}{t_1\,t_2\ldots t_{n-1}}\right)\left(\frac{xz\,T_1T_2\ldots T_{n-1}}{t_1t_2\ldots t_{n-1}}\right)J_{n_n}(pz)(pz)^{\frac{1}{2}}f(p)\,dp\,dz.$$

$$=T_1\,T_2\ldots T_{n-1}\int_0^\infty J_{n_n}(zv)\,(zv)^{\frac{1}{2}}\,dz\int_0^\infty f(p)\,J_{n_n}(pz)\,(pz)^{\frac{1}{2}}\,dp$$

$$=T_1\,T_2\ldots T_{n-1}f\left(\frac{xT_1\,T_2\ldots T_{n-1}}{t_1\,t_2\ldots t_{n-1}}\right),\text{ by Hankel's theorem}$$

where, $\quad v=\left(\frac{x\,T_1\,T_2\ldots T_{n-1}}{t_1\,t_2\ldots t_{n-1}}\right).$

Hence

$$I=\int_0^\infty\int_0^\infty\ldots\int_0^\infty J_{n_1}(t_1)\,J_{n_2}(t_2)\ldots J_{n_{n-1}}(t_{n-1})\,J_{n_1}(T_1)\ldots J_{n_{n-1}}(T_{n-1})$$

$$\times\left(\frac{T_1\,T_2\ldots T_{n-1}}{t_1\,t_2\ldots t_{n-1}}\right)^{\frac{1}{2}}f\left(\frac{x\,T_1\,T_2\ldots T_{n-1}}{t_1\,t_2\ldots t_{n-1}}\right)dt_1\,dt_2\ldots dt_{n-1}dT_1dT_2\ldots dT_{n-1}.$$

Again

$$\int_0^\infty\int_0^\infty J_{n_{n-1}}(t_{n-1})\,J_{n_{n-1}}(T_{n-1})(T_{n-1}t_{n-1})^{\frac{1}{2}}\,f\left(\frac{xT_1T_2\ldots T_{n-1}}{t_1\,t_2\ldots t_{n-1}}\right)\frac{dt_{n-1}\,dT_{n-1}}{t_{n-1}}$$

$$=\int_0^\infty J_{n_{n-1}}\left(\frac{x\,T_1\,T_2\ldots T_{n-2}\,t_{n-1}}{t_1\,t_2\ldots t_{n-1}}\right)\left(\frac{t_{n-1}xT_1T_2\ldots T_{n-2}}{t_1\,t_2\ldots t_{n-1}}\right)^{\frac{1}{2}}dt_{n-1}$$

$$\times\int_0^\infty J_{n_{n-1}}(T_{n-1}\,t_{n-1})(T_{n-1}\,t_{n-1})^{\frac{1}{2}}f\;\;(T_{n-1})dT_{n-1};$$

now by writing $\frac{x\,T_1\,T_2\ldots T_{n-2}}{t_1\,t_2\ldots t_{n-2}}$ for t_{n-1} and changing the order of integrations and then writing $T_{n-1}\,t_{n-1}$ for T_{n-1} and assuming that changes in the ord·rs of integration are permissible, we get

$$=f\left(\frac{x\,T_1\,T_2\ldots T_{n-2}}{t_1\,t_2\ldots t_{n-2}}\right)$$

$$I=\int_0^\infty\int_0^\infty\ldots\int_0^\infty J_{n_1}(t_1)J_{n_1}(T_1)J_{n_2}(t_2)\ldots J_{n_{n-2}}(t_{n-2})J_{n_{n-2}}(T_{n-2})\left(\frac{T_1\,T_2\ldots T_{n-2}}{t_1\,t_2\ldots t_{n-2}}\right)^{\frac{1}{2}}$$

$$\times\,f\left(\frac{x\,T_1\,T_2\ldots T_{n-2}}{t_1\,t_2\ldots t_{n-2}}\right)dt_1\,dt_2\ldots dt_{n-2}dT_1\,dT_2\ldots dT_{n-2}$$

$$\int_0^\infty \int_0^\infty J_{n_{n-2}}(t_{n-2}) J_{n_{n-2}}(T_{n-2})(T_{n-2}t_{n-2})^{\frac{1}{2}} f\left(\frac{x\,T_1\,T_2\ldots T_{n-2}}{t_1\,t_2\ldots t_{n-2}}\right)\frac{dt_{n-2}}{t_{n-2}}\,dT_{n-2}$$

$$= \int_0^\infty J_{n_{n-2}}\left(\frac{x\,T_1\,T_2\ldots T_{n-3}\,t_{n-2}}{t_1\,t_2\ldots t_{n-3}}\right)\left(t_{n-2}\,\frac{x\,T_1\,T_2\ldots T_{n-3}}{t_1\,t_2\ldots t_{n-3}}\right)^{\frac{1}{2}} dt_{n-2}$$

$$\times \int_0^\infty J_{n_{n-2}}(T_{n-2}t_{n-2})(t_{n-2}T_{n-2})^{\frac{1}{2}} f(T_{n-2})\,dT_{n-2}$$

$$= f\left(\frac{x\,T_1\,T_2\ldots T_{n-3}}{t_1\,t_2\ldots t_{n-3}}\right)$$

and proceeding exactly as before, we have ultimately

$$I = \int^\infty \int_0^\infty J_{n_1}(t_1)\,J_{n_1}(T_1)\,f\left(\frac{xT_1}{t_1}\right)(T_1\,t_1)^{\frac{1}{2}}\,\frac{dt_1}{t_1}\quad dt_1$$

$$= \int_0^\infty J_{n_1}(xt_1)(xt_1)^{\frac{1}{2}}\,dt_1 \int_0^\infty (T_1\,t_1)^{\frac{1}{2}} f(T_1)\,J_{n_1}(T_1 t_1)\,dT_1$$

$= f(x)$, formally, by Hankel's formula.

Consequently $f(x)$ and $g(x)$ are formally-reciprocal functions in the transform with the kernel $\varpi_{n_1, n_2 \ldots n_n}(xy)$. When $g(x) = f(x)$, the function becomes self-reciprocal.

3. We shall now obtain certain solutions of the problem by taking the generalised transform with the kernel $\varpi_{n_1, n_2 \ldots n_n}(xy)$.

Let $\tau(s)$ be the Mellin transform of $f(x)$. We have the theorem.

Let $x^{k-1} f(x)$ belong to $L(0, \infty)$ and let $f(x)$ be continuous at x.

If
$$\tau(s) = \int_0^\infty x^{s-1} f(x)\,dx, \qquad (s = k + it)$$

then
$$f(x) = \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} \tau(s)\,x^{-s}\,ds.$$

Let $\tau(k+iu)$ belong to $L(-\infty, \infty)$ and let it be continuous in the neighbourhood of the point $u = t$,

Let
$$f(x) = \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} \tau(s)\,x^{-s}\,ds$$

then
$$\tau(k+it) = \int_0^\infty f(x)\,x^{k+it-1}$$

Let
$$f(x) = \int_0^\infty \varpi_{n_1, n_2, \ldots n_n}(xy)\,f(y)\,dy.$$

Hence $$\tau(s) = \int_0^\infty x^{s-1}\,dx \int_0^\infty \varpi_{n_1, n_2,\ldots n_n}(xy) f(y)\,dy$$

$$= \int_0^\infty f(y)\,dy \int_0^\infty x^{s-1}\,\varpi_{n_1, n_2,\ldots n_n}(xy)\,dx$$

which after considerable reductions becomes

$$= 2^{ns-\frac{1}{2}n} \frac{\Gamma(\frac{1}{2}n_1+\frac{1}{2}s+\frac{1}{4})\,\Gamma(\frac{1}{2}n_2+\frac{1}{2}s+\frac{1}{4})\ldots\Gamma(\frac{1}{2}n_n+\frac{1}{2}s+\frac{1}{4})}{\Gamma(\frac{1}{2}n_1-\frac{1}{2}s+\frac{3}{4})\,\Gamma(\frac{1}{2}n_2-\frac{1}{2}s+\frac{3}{4})\,\Gamma(\frac{1}{2}n_n-\frac{1}{2}s+\frac{3}{4})}\,\tau(1-s).$$

Let $$\tau(s) = 2^{\frac{1}{2}ns}\Gamma(\tfrac{1}{2}n_1+\tfrac{1}{2}s+\tfrac{1}{4})\,\Gamma(\tfrac{1}{2}n_2+\tfrac{1}{2}s+\tfrac{1}{4})\ ..\Gamma(\tfrac{1}{2}n_n+\tfrac{1}{2}s+\tfrac{1}{4})\,\psi(s)$$

then $$\psi(s) = \psi(1-s).$$

We thus obtain a general solution, *viz.*

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}ns}\,\Gamma(\tfrac{1}{2}n_1+\tfrac{1}{2}s+\tfrac{1}{4})\,\Gamma(\tfrac{1}{2}n_2+\tfrac{1}{4})\ldots\Gamma(\tfrac{1}{2}n_n+\tfrac{1}{2}s+\tfrac{1}{4})\,\psi(s)\,x^{-s}ds \tag{1}$$

where $\psi(s)$ satisfies the above relation and c is any value of k in the strip $a < k < 1-a$.

Theorem 1. This theorem is analogons to the theorem proved by us in the previous paper, [Bhatnagar, 1954 (1]

We shall say that $f(x)$ belongs to $A(w, a)$ where $0 < \omega \leq \pi$, $a < \frac{1}{2}$, if it is analytic and regular in the region defined by $r > 0$, $|\theta| < \omega$ and if it is $O\,[|x|^{-a-\epsilon}]$ for small x, and $O[|x|^{a-1+\epsilon}]$ for large x, for every positive ϵ and uniformly in the angle $|\theta| \leq \omega-\eta<\omega$.

The necessary and sufficient condition that $f(x)$ of $A(w, a)$ shall be its own $R_{n_1, n_2 \ldots n_n}$ transform is that it should be of the form (1), where $\psi(s)$ is regular and satisfies the relation $\psi(s) = \psi(1-s)$ in the strip $a < k < 1-a$ and is of the order $O\,[e^{(\frac{1}{4}n\pi-\omega+\mu)|t|}]$, $|\theta| \geq \omega-\mu < \omega$.

The proof is on the same lines as that in Titchmarsh, (1948).

4. The expression of $\varpi_{n_1, n_2 \ldots n_n}(x)$ in the form of hypergeometric functions.

By the Mellin's Inversion formula, we have (Titchmarsh, 1948)

$$\varpi_{n_1, n_2 \ldots n_n}(x) = \frac{1}{2\pi}\int_{c-i\infty}^{c+i\infty} R(s)\,x^{-s}ds$$

where $$R(s) = 2^{ns-\frac{1}{2}n}\frac{\Gamma(\frac{1}{2}n_1+\frac{1}{2}s+\frac{1}{4})\,\Gamma(\frac{1}{2}n_2+\frac{1}{2}s+\frac{1}{4})\ ..\Gamma(\frac{1}{2}n_n+\frac{1}{2}s+\frac{1}{4})}{\Gamma(\frac{1}{2}n_1-\frac{1}{2}s+\frac{3}{4})\,\Gamma(\frac{1}{2}n_2-\frac{1}{2}s+\frac{3}{4})\ldots\Gamma(\frac{1}{2}n_n-\frac{1}{2}s+\frac{3}{4})}$$

so that $$\varpi_{n_1, n_2,\ldots n_n}(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{ns-\frac{1}{2}n}\,\frac{\Gamma(\frac{1}{2}n_1+\frac{1}{2}s+\frac{1}{4})\ldots\Gamma(\frac{1}{2}n_n+\frac{1}{2}s+\frac{1}{4})}{\Gamma(\frac{1}{2}n_1-\frac{1}{2}s+\frac{3}{4})\ldots\Gamma(\frac{1}{2}n_n-\frac{1}{2}s+\frac{3}{4})}\,x^{-s}ds$$

Hence, by the theory of Residues,

$$\varpi_{n_1,n_2,\ldots n_n}(x)=\left\{\begin{aligned}&\frac{x^{n_1+\frac12}}{2^{(n-1)+n_1}}\,\frac{\Gamma(\frac12 n_n-\frac12 n_1)\Gamma(\frac12 n_{n-1}-\frac12 n_1)\ldots\Gamma(\frac12 n_2-\frac12 n_1)}{\Gamma(n_1+1)\,\Gamma(\frac12 n_1+\frac12 n_2+1)\ldots\Gamma(\frac12 n_1+\frac12 n_n+1)}\times\\&\times{}_0F_{2n-1}\left[\begin{matrix}n_1+1,\ \frac12 n_1+\frac12 n_2+1,\ \frac12 n_1+\frac12 n_3+1,\ \ldots\frac12 n_1+\frac12 n_n+1\\ \frac12 n_1-\frac12 n_2+1,\ \frac12 n_1-\frac12 n_3+1,\ldots\frac12 n_1-\frac12 n_n+1;(-1)^n x^2 2^{2n}\end{matrix}\right]+\\&\qquad+\ \ldots\ldots\ +\ \ldots\ldots\ +\ \ldots\\&\qquad+\ \ldots\ldots\\&+\frac{x^{n_n+\frac12}}{2^{(n-1)+n_n}}\,\frac{\Gamma(\frac12 n_1-\frac12 n_n)\Gamma(\frac12 n_2-\frac12 n_n)\ldots\Gamma(\frac12 n_{n-1}-\frac12 n_n)}{\Gamma(n_n+1)\Gamma(\frac12 n_n+n_1+1)\ldots\Gamma(\frac12 n_n+n_{n-1}+1)}\times\\&\times{}_0F_{2n-1}\left[\begin{matrix}n_n+1,\ \frac12 n_n+n_{n_1}+1,\ldots\frac12 n_n+n_{n-1}+1\\ \frac12 n_n-\frac12 n_1+1,\ \frac12 n_n-\frac12 n_2+1,\ldots\frac12 n_n-\frac12 n_{n-1}+1;\ (-1)^n x^2 2^{2n}\end{matrix}\right]\end{aligned}\right\}$$

In particular,

(i) $$\varpi_{\mu,\nu,\lambda}(x)=\left\{\begin{aligned}&\frac{x^{\mu+\frac12}}{2^{2+3\mu}}\,\frac{\Gamma(\frac12\lambda-\frac12\mu)\,\Gamma(\frac12\nu-\frac12\mu)}{\Gamma(\mu+1)\,\Gamma(\frac12\mu+\frac12\nu+1)\,\Gamma(\frac12\mu+\frac12\lambda+1)}\times\\&\qquad{}_0F_5\left[\begin{matrix}\mu+1,\ \frac12\mu+\frac12\nu+1,\ \frac12\mu+\frac12\lambda+1\\ \frac12\mu-\frac12\nu+1,\ \frac12\mu-\frac12\lambda+1;\ -x^2/64\end{matrix}\right]\\&+\frac{x^{\nu+\frac12}}{2^{2+3\nu}}\,\frac{\Gamma(\frac12\lambda-\frac12\nu)\,\Gamma(\frac12\mu-\frac12\nu)}{\Gamma(\nu+1)\,\Gamma(\frac12\nu+\frac12\mu+1)\,\Gamma(\frac12\nu+\frac12\lambda+1)}\\&\qquad{}_0F_5\left[\begin{matrix}\nu+1,\ \frac12\nu+\frac12\mu+1,\ \frac12\nu+\frac12\lambda+1\\ \frac12\nu-\frac12\mu+1,\ \frac12\nu-\frac12\lambda+1;\ -x^2/64\end{matrix}\right]\\&+\frac{x^{\lambda+\frac12}}{2^{2+3\lambda}}\,\frac{\Gamma(\frac12\mu-\frac12\lambda)\,\Gamma(\frac12\nu-\frac12\lambda)}{\Gamma(\lambda+1)\,\Gamma(\frac12\mu+\frac12\lambda+1)\Gamma(\frac12\lambda+\frac12\nu+1)}\\&\qquad{}_0F_5\left[\begin{matrix}\lambda+1,\ \frac12\lambda+\frac12\mu+1,\ \frac12\lambda+\frac12\nu+1\\ \frac12\lambda-\frac12\mu+1,\ \frac12\lambda-\frac12\nu+1;\ -x^2/64\end{matrix}\right]\end{aligned}\right\}$$

(ii) $$\varpi_{\mu,\nu}(x)=\left\{\begin{aligned}&\frac{x^{\mu+\frac12}}{2^{1+2\mu}}\,\frac{\Gamma(\frac12\nu-\frac12\mu)}{\Gamma(\mu+1)\Gamma(\frac12\mu+\frac12\nu+1)}\,{}_0F_3\left[\mu+1,\ \frac12\mu+\frac12\nu+1,\ \frac12\mu-\frac12\nu+1;\ x^2/16\right]\\&+\frac{x^{\nu+\frac12}}{2^{1+2\nu}}\,\frac{\Gamma(\frac12\mu-\frac12\nu)}{\Gamma(\nu+1)\,\Gamma(\frac12\nu+\frac12\mu+1)}\,{}_0F_3\left[\nu+1,\ \frac12\nu+\frac12\mu+1,\ \frac12\nu-\frac12\mu+1;\ x^2/16\right]\end{aligned}\right\}$$

(iii) $$x^{\frac12}J_\mu(x)=\frac{x^{\mu+\frac12}}{2^\mu\,\Gamma(\mu+1)}\,{}_0F_1\left[\mu+1;\ \tfrac14 x^2\right]$$

5. **Theorem 2.** *Let*

$$\psi(p)=\int_0^\infty\frac{t^{\nu+3/2}f(t)\,dt}{(t^2+p^2)^{\nu+3/2}},\qquad \nu\geq-\tfrac12$$

also let $f(t)$ *be continuous, bounded and absolutely integrable in* $(0, \infty)$. *Further let* $\int_0^1 f(y)/y\, dy$ *exist and*

$$k(px) = (px)^{\nu+\frac{1}{2}} \int_0^\infty \frac{e^{-ptx} t^{2\nu+1}\, dt}{(1+t^2)^{\nu+3/2}} = (px)^{\nu+\frac{1}{2}} \int_0^{\frac{1}{2}\pi} \exp(-px \tan\theta),\ \sin^{2\nu+1}\theta d\theta$$

also
$$\chi_1(p) = \int_0^\infty f(x)\, K(px)\, dx \tag{2}$$

Then if $f(t)$ *is* $R_{\nu+1}$, $\chi_1(t)$ *is also* $R_{\nu+1}$. When $\nu = -\frac{1}{2}$ the function $K(px)$ becomes a particular case of Gilbert's function (Erdelyi, 1937),

We have seen (Bhatnagar, 1954 (3)) that if $f(t)$ is $R_{\nu+1}$, then $p_{\nu+\frac{1}{2}}\psi(p)$ is R_ν.

Let
$$\chi_1(p) = p^{\nu+\frac{1}{2}} \int_0^\infty e^{-pt} t^{2\nu+1} \psi(t)\, dt.$$

By a theorem due to Mitra and Bose, (1953), since $t^{\nu+\frac{1}{2}}\psi(t)$ is R_ν, $\chi_1(p)$ is $R_{\nu+1}$ provided $\chi_1(p)/p^{\nu+\frac{1}{2}}$ is continuous in $(0, \infty)$.

Now
$$\chi_1(p) = p^{\nu+\frac{1}{2}} \int_0^\infty e^{-pt} t^{2\nu+1} dt \int_0^\infty \frac{x^{\nu+3/2} f(x)\, dx}{(x^2+t^2)^{\nu+3/2}}$$

$$= p^{\nu+\frac{1}{2}} \int_0^\infty x^{\nu+3/2} f(x)\, dx \int_0^\infty \frac{e^{-pt}\, t^{2\nu+1}\, dt}{(x^2+t^2)^{\nu+3/2}}$$

$$= \int_0^\infty K(px)\, f(x)\, dx, \text{ which is our relation (2).}$$

Hence if $f(t)$ is $R_{\nu+1}$, $\chi_1(t)$ is also $R_{\nu+1}$ provided $\chi_1(p)/p^{\nu+\frac{1}{2}}$ is continuous in $(0, \infty)$ and $\chi_1(p)$ is defined as above.

If,
$$\chi_2(t) = \int_0^\infty \chi_1(t)\, K(tx)\, dt$$

then $\chi_2(t)$ is also $R_{\nu+1}$, provided $\chi_2(t)/t^{\nu+\frac{1}{2}}$ is continuous in $(0, \infty)$ and so on. The converse also holds good.

We shall next show that $\chi(t)$ defined as above is $R_{\nu+1,\nu+1}$ *i.e* self-reciprocal in $\varpi_{\mu,\nu}$ transform.

We have

$$\int_0^\infty K(px)\, \varpi_{\nu+1,\nu+1}(xy)\, dx$$

$$= \int_0^\infty \frac{t^{2\nu+1}\, dt}{(1+t^2)^{\nu+3/2}} \int_0^\infty (px)^{\nu+\frac{1}{2}} e^{-ptx}\, \varpi_{\nu+1,\nu+1}(xy)\, dx$$

$$= p^{\nu+\frac{1}{2}}\, y^{\frac{1}{2}} \int_0^\infty \frac{t^{2\nu+1}}{(1+t^2)^{\nu+3/2}} \left\{ \int_0^\infty x^{\nu+1} e^{-ptx}\, dx \int_0^\infty J_{\nu+1}(T)\, J_{\nu+1}\left(\frac{xy}{T}\right) \frac{dT}{T} \right\}$$

$$= \frac{2^{\nu+1}\Gamma(\nu+3/2)}{\pi^{\frac{1}{2}}}\, y(py)^{\nu+\frac{1}{2}} \int_0^\infty \frac{t^{2\nu+1}\, dt}{(1+t^2)^{\nu+3/2}} \int_0^\infty \frac{J_{\nu+1}(T)\, T^{\nu+1}\, dT}{(p^2 t^2 T^2 + y^2)^{\nu+3/2}}$$

$$= \frac{2^{\nu+1}\Gamma(\nu+3/2)}{\pi^{\frac{1}{2}}}(py)^{\nu+3/2} \int_0^\infty J_{\nu+1}(T) T^{\nu+2} dT \int_0^\infty \frac{t^{2\nu+1}\, dt}{(t^2+y^2)^{\nu+3/2}(t^2+p^2T^2)^{\nu+3/2}}$$

on changing the order of integration and writing $\dfrac{t}{pT}$ for t,

$$= \frac{2^{\nu+1}\Gamma(\nu+3/2)}{\pi^{\frac{1}{2}}}\left(\frac{y}{p}\right)^{\nu+3/2} \int_0^\infty \frac{t^{2\nu+1}\, dt}{(t^2+y^2)^{\nu+3/2}} \int_0^\infty \frac{J_{\nu+1}(T)\, T^{\nu+2} dT}{(T^2+t^2/p^2)^{\nu+3/2}}$$

$$= \left(\frac{y}{p}\right)^{\nu+3/2} \int_0^\infty \frac{e^{-t/p}\, t^{2\nu+1}\, dt}{(t^2+y^2)^{\nu+3/2}}$$

$$= \frac{1}{p}\left(\frac{y}{p}\right)^{\nu+\frac{1}{2}} \int_0^\infty \frac{t^{2\nu+1}\, e^{-yt/p}\, dt}{(1+t^2)^{\nu+3/2}}$$

$$= \frac{1}{p}\, K\left(\frac{y}{p}\right).$$

Hence $\displaystyle\int_0^\infty K(x)\, \varpi_{\nu+1,\nu+1}(x)\, dx = K(y)$, *i.e.*, $K(x)$ is $R_{\nu+1,\nu+1}$

Therefore $\displaystyle\chi_1(p) = \int_0^\infty K(px)\, dx \int_0^\infty f(y)\, \varpi_{\nu+1,\nu+1}(xy)\, dy$; if $f(x)$ is $R_{\nu+1,\nu+1}$

$$= \int_0^\infty f(y)\, dy \int_0^\infty K(px)\, \varpi_{\nu+1,\nu+1}(xy)\, dx$$

$$= \frac{1}{p}\int_0^\infty K\left(\frac{y}{p}\right) f(y)\, dy$$

$$= \frac{1}{p}\chi_1\left(\frac{1}{p}\right)$$

This also follows by a theorem proved before [Bhatnagar 1954, (1)]. The various changes in the orders of integrations are justifiable.

Again, let $f(t)$ be $R_{\nu+1,\nu+1,\nu+1}$. We have just now seen that

$$\int_0^\infty K(px)\, \varpi_{\nu+1,\nu+1}(xy)\, dx = \frac{1}{p}\, K\left(\frac{y}{p}\right)$$

Also
$$\chi_1(p) = \int_0^\infty K(px)\, f(x)\, dx$$
$$= \int_0^\infty K(px)\, dx \int_0^\infty f(y)\, \varpi_{\nu+1,\,\nu+1,\,\nu+1}(xy)\, dy$$
$$= \int_0^\infty f(y)\, dy \left\{ \int_0^\infty J_{\nu+1}(t) \frac{dt}{t^{\frac12}} \int^\infty K(px)\, \varpi_{\nu+1,\nu+1}\left(\frac{xy}{t}\right) dx \right\}$$
$$= \frac{1}{p} \int_0^\infty f(y)\, dy \int_0^\infty J_{\nu+1}(t)\, K\left(\frac{y}{pt}\right) \frac{dt}{t^{\frac12}} \quad \ldots$$
$$= \int_0^\infty t^{\frac12} J_{\nu+1}(t) \frac{1}{pt} \chi_1\left(\frac{1}{pt}\right) dt$$
or
$$\frac{1}{p}\chi_1\left(\frac{1}{p}\right) = \int_0^\infty (pt)^{\frac12} J_{\nu+1}(pt)\, \chi_1\left(\frac{1}{t}\right) \frac{dt}{t}$$

which shows that $\frac{1}{t}\chi_1\left(\frac{1}{t}\right)$ is $R_{\nu+1}$, provided the integral for the self-reciprocal property exists and the changes in the orders of integrations are permissible.

Example. Let $f(x)$ be $e^{-\frac12 x^2} x^{\nu+3/2}$ which is $R_{\nu+1}$.
$$\chi(p) = p \int_0^\infty e^{-pt}\, t^{2\nu+1}\, dt \int_0^\infty \frac{e^{-\frac12 x^2} x^{2\nu+3}\, dx}{(x^2+t^2)^{\nu+3/2}}.$$

Putting $\frac12 x^2 = z$ and then $z = \frac12 t^2 y$, we get after a little simplification,
$$\chi(p) = \tfrac12 p \int_0^\infty e^{-pt}\, t^{2\nu+2}\, dt \int_0^\infty e^{-\frac12 t^2 y} \frac{y^{\nu+1}\, dy}{(1+y)^{\nu+3/2}}.$$

But
$$\int_0^\infty \frac{e^{-pt}\, t^{\nu+1}\, dt}{(1+t)^{\nu+3/2}} = 2^{\nu+3/2} \frac{\Gamma(\nu+2)}{p^{\frac12}}\, e^{\frac12 p}\, D_{-2\nu-3}\left((2p)^{\frac12}\right).$$

Hence $\chi(p) = A\, p \int_0^\infty e^{-pt} t^{2\nu+1} e^{\frac14 t^2} D_{-2\nu-3}(t)\, dt$, where A is a constant independent of p.

It can directly be verified that $p^{\nu-\frac12}\chi(p)$ is $R_{\nu+1}$.

Corollary. Let
$$\phi(p) = \int_0^\infty K(px)\, \varpi_{\nu+1,\nu+1,\nu+1}(x), \qquad (\nu > -\tfrac12)$$
$$= \int^\infty K(px)\, dx \int_0^\infty \frac{J_{\nu+1}(t)}{t^{\frac12}}\, \varpi_{\nu+1,\nu+1}\left(\frac{x}{t}\right) dt$$

$$= \frac{1}{p}\int_0^\infty J_{\nu+1}(t)\,\frac{dt}{t^{\frac{1}{2}}}\int_0^\infty K(x)\,\varpi_{\nu+1,\nu+1}\left(\frac{x}{pt}\right)dx$$

$$= \int_0^\infty t^{\frac{1}{2}} J_{\nu+1}(t)\,K\left(\frac{1}{pt}\right)\frac{dt}{pt}.$$

Therefore $$\frac{1}{p}\varphi\left(\frac{1}{p}\right) = \int_0^\infty t^{\frac{1}{2}} J_{\nu+1}(t)\,K\left(\frac{1}{p}\right)\frac{dt}{t}$$

Since $t^{\frac{1}{2}} J_{\nu+1}(t)$ is $R_{\nu+1,\nu+1}$ and $K(t)$ is also $R_{\nu+1,\nu+1}$; $\frac{1}{p}\varphi\left(\frac{1}{p}\right)$ is $R_{\nu+1,\nu+1,\nu+1,\nu+1}$ provided the integral for the self-reciprocal property exists [Bhatnagar, 1953 (2)].

6. We shall now simply state a theorem on a function of two variables being self-reciprocal in the $\varpi_{\mu,\nu}$ transform. We do not give the proof here as it is long and is almost on the same lines as for a function of single variable given in one of the earlier papers [Bhatnagar, 1954 (1)].

Theorem 3. *The necessary and sufficient conditions that $f(t_1, t_2)$* shall be self-reciprocal in $\varpi_{\mu,\nu}$ transform is that*

$$\varphi_1\left(\frac{1}{p_1}, \frac{1}{p_2}\right) = p_1^{2n_1+2}\,p_2^{2n_2+2}\,\varphi_1(p_1, p_2).$$

i.e., $f(t_1, t_2)$ is $R_{(n_1+s_1-m_1,\, n_1+s_1+m_1),\ (n_2+s_2-m_2,\, n_2+s_2+m_2)}$

where $$\varphi_1(p_1, p_2) = \frac{2}{\pi}\int_0^\infty\int_0^\infty (p_1t_1)^{s_1}(p_2t_2)^{s_2}t_1^{n_1+\frac{1}{2}}t_2^{n_2+\frac{1}{2}}K_{m_1}(p_1t_1)K_{m_2}(p_2t_2)f(t_1t_2)\,dt_1dt_2$$

An example on Two Variables. Let

$$f(t_1, t_2) = \frac{(t_1\,t_2)^{\frac{1}{2}}}{t_1^2+t_2^2}\,; \quad \begin{array}{l} s_1 = s_2 = 0,\ m_1 = m_2 0, \\ n_1 = n_2 = 1. \end{array}$$

then $$\varphi_1(p_1, p_2) = \frac{2}{\pi}\int_0^\infty\int_0^\infty (t_1\,t_2)^{3/2}K_0(p_1\,t_1)\,K_0(p_2\,t_2)\,\frac{(t_1\,t_2)^{\frac{1}{2}}}{t_1^2+t_2^2}\,dt_1\,dt_2.$$

Therefore $$\varphi_1(p_1, p_2) = -\frac{4(p_1^2-p_2^2)}{(p_1^2+p_2^2)^3}\log\frac{p_1}{p_2} + \frac{4}{(p_1^2+p_2^2)^2} +$$

$$+\frac{\pi}{2}\,\frac{(p_1^4-6p_1^2\,p_2^2+p_2^4)}{p_1\,p_2(p_3^2+p_2^2)^3}\,;$$

so that $$\varphi_1(p_1\,p_2) = (p_3\,p_2)^{-1}\varphi_1\left(\frac{1}{p_1}, \frac{1}{p_2}\right).$$

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* $f(t_1, t_2)$ is continuous, bounded and absolutely integrable in $(0, \infty)$. The conditions are only sufficient but not necessary.

Hence $f(t_1, t_2)$ is self-reciprocal in the $\varpi_{\mu,\nu}$ transform as can be verified directly also. We might note here that by suitable choice of s_1, s_2, m_1, m_2 and n_1, n_2 we fall back upon some of the results of Erdelyi (1937).

I am thankful to Dr. S. C. Mitra for his help and guidance in the preparation of this paper.

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AND STATISTICS
LUCKNOW UNIVERSITY, LUCKNOW.

References

Bhatnagar, K. P. (1954), (1), *Bull Cal. Math. Soc.*, **46**, 179.

———, (1953), (2), *Bulletin de la classe des Sciences de l'Academie Royale de Belgique*, **39**, 42.

———, (1954), (3) *Bull. Cal. Math. Soc.* **46**, 251

Erdelyi, A., (1937), *Atti della Reale Accademia della Scienze di Torino*, **72**.

McLachlan, N. W. & Humbert, P., (1947), *Formulaire pour le calcul symbolique* Memorial des Sciences Mathemtiques.

Mitra, S. C. & Bose, B N, (1953), *Acta Mathematica*, **87**, 1.

Titchmarsh, E. C., (1948), *The theory of Fourier Integrals*.

Watson, G. N., (1931), *Quart. Journ, of Math.* (Oxford), **2**, 298.

ON THE SUMMABILITY $|C, 1|$ OF FOURIER SERIES.

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(*Received—September 23, 1953.*)

1. We suppose that $f(t)$ is integrable L in the interval $(-\pi, \pi)$, and periodic with period 2π, and we write

$$\varphi(t)=\tfrac{1}{2}\{f(x+t)+f(x-t)\}. \tag{1}$$

We suppose as we may that the Fourier series of $\varphi(t)$ is $\Sigma A_n \cos nt$, where $A_0=0$, so that Fourier series of $f(t)$ at the point $t=x$ is ΣA_n, where $A_n=a_n \cos nx+b_n \sin nx$.

It is known (Bosanquet and Kestelman, 1939) that the summability $|C, 1|$ of the Fourier series of $f(t)$ at a point $t=x$ is not a local property but depends on the behaviour of $f(t)$ in the whole interval $(-\pi, \pi)$. The object of this note is to show that, if

$$\Sigma |A_n| n^{-1} \log n < \infty, \tag{2}$$

then the summability $|C, 1|$ of the Fourier series depends only on the properties of $f(t)$ near the point $t=x$.

We denote the n-th Cesàro means of order 1 of the sequences $S_n=A_1+A_2+\ldots+A_n$, and $t_n=nA_n$ by σ_n and τ_n respectively. The series ΣA_n is said to be summable $|C, 1|$, if the series $\Sigma|\sigma_n-\sigma_{n-1}|$ is convergent.

From the identity

$$n(\sigma_n-\sigma_{n-1})=\tau_n$$

it follows that the summability $|C, 1|$ of ΣA_n is equivalent to the convergence of $\Sigma n^{-1}|\tau_n|$.

2. Now

$$\begin{aligned}\tau_n &= (A_1+2A_2+\ldots+nA_n)/n \\ &= \frac{2}{\pi n}\int_0^\pi \varphi(t)\sum_{m=1}^{n} m \cos mt \, dt \\ &= \frac{2}{\pi n}\int_0^\pi \varphi(t)\frac{d}{dt}\Big(\sum_{m=1}^{n} \sin mt\Big)\, dt\end{aligned}$$

$$=\frac{2}{\pi n}\int_0^\pi \varphi(t)\Big\{\tfrac{1}{4}\operatorname{cosec}^2 \tfrac{1}{2}t \cos \tfrac{1}{2}t \cos (n+\tfrac{1}{2})t-\tfrac{1}{4}\operatorname{cosec}^2\tfrac{1}{2}t+(n+\tfrac{1}{2})\,\frac{\sin (n+\tfrac{1}{2})t}{2 \sin \tfrac{1}{2}t}\Big\}dt$$

$$=\int_0^\pi \varphi(t)g(n, t)\, dt, \text{ say.}$$

Write

$$\tau_n=\int_0^\eta \varphi(t)g(n, t)dt+\frac{1}{2\pi n}\int_\eta^\pi \varphi(t)\cos(n+\tfrac{1}{2})t\cos\tfrac{1}{2}t\,\mathrm{cosec}^2\tfrac{1}{2}t\,dt$$

$$-\frac{1}{2\pi n}\int_\eta^\pi \varphi(t)\,\mathrm{cosec}^2\tfrac{1}{2}t\,dt+\frac{2}{\pi}\frac{n+\frac{1}{2}}{n}\int_\eta^\pi \varphi(t)\frac{\sin(n+\frac{1}{2})t}{2\sin\frac{1}{2}t}dt$$

$$=\int_0^\eta \varphi(t)g(n, t)dt+\alpha_n-\beta_n+\gamma_n, \text{ say, where } 0<\eta<\pi.$$

It is seen atonce that each of α_n and β_n is $O(n^{-1})$ and hence it follows that each of the series

$$\Sigma n^{-1}|\alpha_n| \text{ and } \Sigma n^{-1}|\beta_n|$$

is convergent.

Substituting the Fourier series of $\varphi(t)$ in the integral for γ_n and proceeding as in the proof of the Hardy-Littlewood test for the convergence of the Fourier series (Zygmund, 1935), it can be shown by using (2) that

$$\Sigma n^{-1}|\gamma_n|<A\sum_{n=1}^{\infty} n^{-1}\sum_{k=1}^{\infty}\frac{|A_k|}{|k-n-\frac{1}{2}|}$$

$$=A\sum_{k=1}^{\infty}|A_k|\sum_{n=1}^{\infty}\frac{n^{-1}}{|k-n-\frac{1}{2}|}$$

$$<A\Sigma|A_k|k^{-1}\log k<\infty.$$

Thus it follows that the series $\Sigma n^{-1}|\tau_n|$ will be convergent, if

$$\Sigma n^{-1}\left|\int_0^\eta \varphi(t)g(n, t)dt\right|<\infty,$$

and this shows that the summability $|C, 1$ of ΣA_n depends only on the properties of $f(t)$ near the point $t=x$. An interesting particular case is

If $$A_n=O\{(\log n)^{-2-\delta}\}, \delta>0, \tag{3}$$

then the summalility $C, 1$ of the Fourier series is a local property.

Remark. If $f(t)$ is L in $(-\pi, \pi)$, then A_n tends to zero. On the other hand, A_n does not tend to zero in any definite order whatever the behaviour of the function near the point. This is the only reason for which the summability $|C, 1|$ is a non-local property. In fact, when an order condition of the type given in (3) above is satified, the summability $|C, 1|$ becomes a local property.

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References.

Bosanquet L. S and Kestelman, H. (1939), *Proc. Lond. Math. Soc.*, (2) **45**, 88.
Zygmund, A. (1935) *Trigonometrical Series*, p. 35.

HYPER-DARBOUX LINES ON A SURFACE IN THREE DIMENSIONAL EUCLIDEAN SPACE

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(*Communicated by the Secretary—Received August 25, 1953*)

A union curve of a congruence G is such a curve of a surface which has the property that its osculating plane at all points contains the tangent to the curve of the congruence G through that point. These curves were studied by Springer (1945) and Mishra (1950 A, B). Mishra (1950 C) further defined and studied hyper-asymptotic curves, *i e.* the curves of a surface whose rectifying plane at all points contains a tangent to the curve of the congruence G through that point.

In this paper we observe the curves which have the property that the plane determined by the tangent of a curve and the vector $R_1 n^i + R_2 \dfrac{dR_1}{ds} b^i$ at all points contains the tangent to the curve of the congruence G through that point, n^i and b^i being the components of a unit vector of the principal normal and binormal, R_1 and R_2 the radii of first and second curvatures of the curve in that point.

1. Let the coordinates of the point P of the surface S in three dimensional Euclidean space be given by $x^i = x^i(u^1, u^2)$; $(i=1, 2, 3)$. Then we have for the first fundamental form

$$g_{\alpha\beta} = \sum_i \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^i}{\partial u^\beta} {}^{*} \tag{1}$$

Let further there be given in the space the conguence of the curves and let λ^i be the contravariant components of the unit tangent vector to the curve of the congruence through x^i. This vector can be decomposed into a component tangential and a component normal to the surface S, *i.e* we can write

$$\lambda^i = l^\alpha \frac{\partial x^i}{\partial u^\alpha} + LX^i, \tag{2}$$

where $\dfrac{\partial x^i}{\partial u^\alpha}$ are the components of a vector on the surface, X^i the components of the unit normal vector to the surface and l^α and L the parameters such that

$$\cos\theta = \sum_i \lambda^i X^i = L.$$

$$\sin^2\theta = g_{\alpha\beta} l^\alpha l^\beta$$

provided that θ is the angle between λ^i and X^i.

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* In what follows Latin indices take the values 1, 2, 3 and Greek indices the values 1, 2.

2. The first two Frenet's formulas, for a curve C in the space, have the form

$$\frac{d}{ds}\left(\frac{dx^i}{ds}\right) = \kappa_1 n^i$$

$$\frac{dn^i}{ds} = -\kappa_1 \frac{dx^i}{ds} + \kappa_2 b^i,$$

where $\frac{dx^i}{ds}$ are the components of unit tangent vector to C, n^i and b^i the components of unit vectors of the principal normal and binormal, κ_1 and κ_2 the first and the second curvatures of C Substituting n^i by $\frac{1}{\kappa_1}\frac{d^2x^i}{ds^2}$ in the second equation, we get

$$\frac{d^3x^i}{ds^3} = -\kappa_1^2 \frac{dx^i}{ds} + \frac{d\kappa_1}{ds} n^i + \kappa_1\kappa_2 b^i.$$

Since the unit vectors $\frac{dx^i}{ds}$, n^i and b^i are mutually orthogonal to one another, it is

$$\sum_i \frac{d^3x^i}{ds^3}\left(R_1 n^i \frac{dR_1}{ds} R_2 b^i\right) = R_1 \frac{d\kappa_1}{ds} + R_2 \frac{dR_1}{ds}\kappa_1\kappa_2 = R_1 \frac{d\kappa_1}{ds} + \kappa_1 \frac{dR_1}{ds} = \frac{d}{ds}(R_1\kappa_1) = 0, \tag{3}$$

where $R_1 = 1/\kappa_1$ and $R_2 = 1/\kappa_2$ are the radii of the first and the second curvature of C.

3. Now, let us define hyper-Darboux lines (hyper-D lines). These are the curves on a surface which have the property that the plane determined by the tangent of the curve and the vector $R_1 n^i + R_2 \frac{dR_1}{ds} b^i$ at every point of the curve contains the tangent to the curve of the congruence through that point. Accordingly, the vector with components λ^i, for the hyper-D line can be expressed as a linear combination of $\frac{dx^i}{ds}$ and $R_1 n^i + R_2 \frac{dR_1}{ds} b^i$, *i.e.*

$$\lambda^i = p\left(R_1 n^i + R_2 \frac{dR_1}{ds} b^i\right) + q \frac{\partial x^i}{\partial u^\beta} \frac{du^\beta}{ds},$$

or

$$l^\alpha \frac{\partial x^i}{\partial u^\alpha} + LX^i = p\left(R_1 n^i + R_2 \frac{dR_1}{ds} b^i\right) + q \frac{\partial x^i}{\partial u^\beta} \frac{du^\beta}{ds}. \tag{4}$$

If we multiply (4) by $\frac{d^3x^i}{ds^3}$ and sum with respect to i, by virtue of (3), we obtain

$$l^\alpha \sum_i \frac{\partial x^i}{\partial u^\alpha} \frac{d^3x^i}{ds^3} + L\sum_i X^i \frac{d^3x^i}{ds^3} = q \sum_i \frac{\partial x^i}{\partial u^\beta} \frac{d^3x^i}{ds^3} \frac{du^\beta}{ds},$$

which we can write in the form

$$\sum_i \frac{\partial x^i}{\partial u^\alpha} \frac{d^3x^i}{ds^3}\left(q \frac{du^\alpha}{ds} - l^\alpha\right) = L\Sigma X^i \frac{d^3x^i}{ds^3}. \tag{5}$$

Multiplying (4) by $\frac{dx^i}{ds}$ and summing with respect to i, after substituting the dummy indices, we get

$$l^\beta \sum_i \frac{\partial x^i}{\partial u^\beta} \frac{dx^i}{ds} = q \sum_i \frac{dx^i}{du^\alpha} \frac{dx^i}{ds} \frac{du^\alpha}{ds},$$

$$l^\beta \sum_i \frac{\partial x^i}{du^\beta} \frac{\partial x^i}{\partial u^\gamma} \frac{du^\gamma}{ds} = q \sum_i \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^i}{\partial u^\gamma} \frac{du^\alpha}{ds} \frac{du^\gamma}{ds},$$

or, in consequence of (1),

$$l^\beta g_{\beta\gamma} \frac{du^\gamma}{ds} = q, \tag{6}$$

since $$g_{\alpha\gamma} \frac{du^\alpha}{ds} \frac{du^\gamma}{ds} = 1.$$

Substituting q from (6) in (5), we have

$$\sum_i \frac{\partial x^i}{\partial u^\alpha} \frac{d^3 x^i}{ds^3}\left(l^\beta g_{\beta\gamma} \frac{du^\gamma}{ds} \frac{du^\alpha}{ds} - l^\alpha\right) = L \sum_i X^i \frac{d^3 x^i}{ds} \tag{7}$$

which is the differential equation of hyper-D lines of surface S.

In the special case, when the congruence is normal to surface S, $l^\alpha = 0$, equation (7) reduces to

$$\sum_i X^i \frac{d^3 x^i}{ds^3} = 0,$$

this being the differentiation of D-line of the surface S [Semin 1952, eq. (2-1)].

If the vectors $\frac{\partial x^i}{\partial u^\alpha}$ and $\frac{d^3 x^i}{ds^3}$ are orthogonal, *i.e.*, if

$$\sum_i \frac{\partial x^i}{\partial u^\alpha} \frac{d^3 x^i}{ds^3} = 0$$

equation (7) denotes that, in this case, too, hyper-D curve reduces to D-line.

Conversely, if hyper-D lines are D-lines of the surface, *i.e.* if $\sum_i X^i \frac{d^3 x^i}{ds^3} = 0$, then either $\sum_i \frac{\partial x^i}{\partial u^\alpha} \frac{d^2 x^i}{ds^3} = 0$, or the vector $l^\beta g_{\beta\gamma} \frac{du^\gamma}{ds} \frac{du^\alpha}{ds} - l^\alpha$ is a zero vector.

4. The quantities $\sum_i X^i \frac{d^3 x^i}{ds^3}$ and $\sum_i \frac{\partial x^i}{\partial u^\alpha} \frac{d^3 x^i}{ds^3}$ may by expressed in function of the coefficients of the two fundamental forms of the surface S and their partial derivatives.

Indeed, from

$$\frac{dx^i}{ds} = \frac{\partial x^i}{\partial u^\alpha}\frac{du^\alpha}{ds}$$

it follows that

$$\frac{d^3x^i}{ds^3} = \frac{\partial^3 x^i}{\partial u^\alpha \partial u^\beta \partial u^\gamma}\frac{du^\alpha}{ds}\frac{du^\beta}{ds}\frac{du^\gamma}{ds} + 3\frac{\partial^2 x^i}{\partial u^\alpha \partial^\beta}\frac{d^2 u^\alpha}{ds^2}\frac{du^\beta}{ds} + \frac{\partial x^i}{\partial u^\alpha}\frac{d^3 u^\alpha}{ds^3} \tag{8}$$

and consequently, we must have

$$\sum_i X^i\frac{d^3x^i}{ds^3} = \sum_i X^i\frac{\partial^3 x^i}{\partial u^\alpha \partial u^\beta \partial u^\gamma}\frac{du^\alpha}{ds}\frac{du^\beta}{ds}\frac{du^\gamma}{ds} + 3\sum_i X^i\frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta}\frac{d^2x^\alpha}{ds^3} + \sum_i X^i\frac{\partial x^i}{\partial u^\alpha}\frac{d^3u^\alpha}{ds^3} \tag{9}$$

If $n_{\alpha\beta}$ are the coefficients of the second fundamental form, it is [Eisenhart, 1947, p. 215, eq. (38·13)]

$$d_{\alpha\beta} = \sum_i X^i\frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta}, \tag{10}$$

and equation (9) reduces to

$$\sum_i X^i\frac{d^3x^i}{ds^3} = \sum_i X^i\frac{\partial^3 x^i}{\partial u^\alpha \partial u^\beta \partial u^\gamma}\frac{du^\alpha}{ds}\frac{du^\beta}{ds}\frac{du^\gamma}{ds} + 3d_{\alpha\beta}\frac{d^2u^\alpha}{ds^2}\frac{du^\beta}{ds}, \tag{11}$$

because $\sum_i X^i\frac{\partial x^i}{\partial u^\gamma} = 0$ [Eisenhart, 1947, p. 213, eq. (38·8)].

If we differentiate (10) with respect to u^γ, we get

$$\frac{\partial d_{\alpha\beta}}{\partial u^\gamma} = \sum_i \frac{\partial X^i}{\partial u^\gamma}\frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta} + \sum_i X^i\frac{\partial^3 x^i}{\partial u^\alpha \partial u^\beta \partial u^\gamma}. \tag{12}$$

Since [Eisenhart 1947, p. 217, eq. (38·19)]

$$\frac{\partial X^i}{\partial u^\gamma} = -d_{\gamma\tau}g^{\tau\sigma}\frac{\partial x^i}{\partial u^\sigma},$$

it follows form (12)

$$\sum_i X^i\frac{\partial^3 x^i}{\partial u^\alpha \partial u^\beta \partial u^\gamma} = \frac{\partial d_{\alpha\beta}}{\partial u^\gamma} + d_{\gamma\tau}g^{\tau\sigma}\sum_i\frac{\partial x^i}{\partial u^\sigma}\frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta}.$$

However, [Eisenhart 1947, p. 216, eq. (38·17)]

$$\sum_i\frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta}\frac{\partial x^i}{\partial u^\sigma} = [\alpha\beta, \sigma] \tag{13}$$

where $[\alpha\beta, \sigma]$ are the Christoffel symbols of the first kind in respect to the fundamental tensor $g_{\alpha\beta}$. Thence we can write

$$\sum_i X^i\frac{\partial^3 x^i}{\partial u^\alpha \partial u^\beta}\frac{\partial x^i}{\partial u^\gamma} = \frac{\partial d_{\alpha\beta}}{\partial u^\gamma} + d_{\gamma\tau}\left\{ {\tau \atop \alpha\beta} \right\},$$

where $\left\{ {\tau \atop \alpha\beta} \right\}$ are the Christoffel symbols of the second kind in respect to the fundamental tensor $g_{\alpha\beta}$.

Substituting $\sum_i X^i \dfrac{\partial^3 x^i}{\partial u^\alpha \partial u^\beta \partial u^\gamma}$ from (14) in (11), we get

$$\sum_i X^i \frac{d^3 x^i}{ds^3} = \left(\frac{\partial d_{\alpha\beta}}{\partial u^\gamma} + d_{\gamma\tau}\left\{ {\tau \atop \alpha\beta} \right\}\right)\frac{du^\alpha}{ds}\frac{du^\beta}{ds}\frac{du^\gamma}{ds} + 3d_{\alpha\beta}\frac{d^2u^\alpha}{ds^2}\frac{du^\beta}{ds}.$$

For the quantities $\sum_i \dfrac{\partial x^i}{\partial u^\alpha}\dfrac{d^3 x^i}{ds^3}$, we have, in consequence of (2)

$$\sum_i \frac{\partial x^i}{\partial u^\alpha}\frac{d^3 x^i}{ds^3} = \sum_i \frac{\partial x^i}{\partial u^\alpha}\frac{\partial^3 x^i}{\partial u^\sigma \partial u^\beta \partial u^\gamma}\frac{du^\sigma}{ds}\frac{du^\beta}{ds}\frac{du^\gamma}{ds} + 3\sum_i \frac{\partial x^i}{\partial u^\alpha}\frac{\partial^2 x^i}{du^\sigma du^\beta}\frac{d^2u^\sigma}{ds_2}\frac{du^\beta}{ds}$$
$$+ \sum_i \frac{\partial x^i}{\partial u^\sigma}\frac{\partial x^i}{\partial u^\alpha}\frac{d^3 x^\sigma}{ds^3};$$

or, in consequence of (13) and (1)

$$\sum_i \frac{\partial x^i}{\partial u^\alpha}\frac{d^3 x^i}{ds^3} = \sum_i \frac{\partial x^i}{\partial u^\alpha}\frac{\partial^3 x^i}{du^\sigma du^\beta \partial u^\gamma}\frac{du^\sigma}{ds}\frac{du^\beta}{ds}\frac{du^\gamma}{ds} + 3[\sigma\beta, \alpha]\frac{d^2u^\sigma}{ds^2}\frac{du^\beta}{ds} + g_{\sigma\alpha}\frac{d^3 x^\sigma}{ds^3}.$$

On the other hand [Eisenhart 1947, p. 216, eq. (38·18)]

$$x^i{}_{,\sigma\beta} = d_{\sigma\beta}X^i\ *$$

i.e.
$$\frac{\partial^2 x^i}{\partial u^\sigma \partial u^\beta} = \left\{ {\tau \atop \sigma\beta} \right\}\frac{\partial x^i}{\partial u^\tau} + d_{\sigma\beta}X^i.$$

If these equations be differentiated with respect to u^γ, we obtain

$$\frac{\partial^3 x^i}{\partial u^\sigma \partial u^\beta \partial u^\gamma} = \frac{\partial}{\partial u^\gamma}\left\{ {\tau \atop \sigma\beta} \right\}\frac{\partial x^i}{\partial u^\tau} + \left\{ {\tau \atop \sigma\beta} \right\}\frac{\partial^2 x^i}{\partial u^\tau \partial u^\gamma} + \frac{\partial d_{\alpha\beta}}{\partial u^\gamma}X^i + d_{\alpha\beta}\frac{\partial X^i}{\partial u^\gamma},$$

and then
$$\sum_i \frac{\partial^3 x^i}{\partial u^\sigma \partial u^\beta \partial u^\gamma}\frac{\partial x^i}{\partial u^\alpha} = \frac{\partial}{\partial u^\gamma}\left\{ {\tau \atop \sigma\beta} \right\}\sum_i \frac{\partial x^i}{\partial u^\tau}\frac{\partial x^i}{\partial u^\alpha} + \left\{ {\tau \atop \sigma\beta} \right\}\sum_i \frac{\partial x^i}{\partial u^\alpha}\frac{\partial^2 x^i}{\partial u^\tau \partial u^\gamma}$$
$$+ \frac{\partial d_{\alpha\beta}}{\partial u^\gamma}\sum X^i \frac{\partial x^i}{\partial u^\gamma} + d_{\sigma\beta}\sum_i \frac{\partial X^i}{\partial u^\gamma}\frac{\partial x^i}{\partial u^\alpha}$$

i.e.
$$\sum_i \frac{\partial^3 x^i}{\partial u^\sigma \partial u^\beta \partial u^\gamma}\frac{\partial x^i}{\partial u^\alpha} = g_{\tau\alpha}\frac{\partial}{\partial u^\gamma}\left\{ {\tau \atop \sigma\beta} \right\} + \left\{ {\tau \atop \sigma\beta} \right\}[\tau\gamma, \alpha] - d_{\sigma\beta}d_{\gamma\alpha}.$$

* The indices after comma indicate covariant differentiation.

Substituting these axpressions in (15), we get

$$\sum_i \frac{\partial x^i}{\partial u^\alpha}\frac{d^3x^i}{ds^3} = \left(g_{\tau\alpha}\frac{\partial}{\partial u^\gamma}\left\{ {\tau \atop \sigma\beta} \right\} + \left\{ {\tau \atop \sigma\beta} \right\}[\tau\gamma, \alpha] - d_{\sigma\beta}d_{\gamma\alpha}\right)\frac{du^\sigma}{ds}\frac{du^\beta}{ds}\frac{du^\gamma}{ds} + 3[\sigma\beta, \alpha] \times$$

$$\times \frac{d^2u^\sigma}{ds^2}\frac{du^\beta}{ds} + g_{\sigma\alpha}\frac{d^3u^\sigma}{ds^3}$$

INSTITUTE OF MATHEMATICS,
ACADEMY SERB OF SCIENCES,
BELGRADE.

References

Eisenhart, L. P. (1947), *Introduction to Differential Geometry with use of Tensor Calculus*, Princeton Univ.

Mishra, R. S. (1950 A), *Math. Student*, **18**, 87.

———— (1950 B), *Bull. Cal. Math. Soc.*, **42**, 218.

———— (1950 C), *Bull. Cal. Math. Soc.*, **42**, 113.

Semin, F. (1952), *Rev. Fac. Sciences Univ., Istambul*, **17**, 351

Springer, C. E. (1945), *Bull. Amer. Math. Soc.*, **51**, 686

Springer, C. E. (1947). *Amer. Math. Monthly*, **54**, 256

ANNUAL REPORT
&
BALANCE SHEET
1954

Calcutta Mathematical Society

CALCUTTA MATHEMATICAL SOCIETY

Report of the Council for the year 1954 to the Annual General Meeting of the Society

The Council of the Calcutta Mathematical Society has the pleasure to submit the following report on the general concerns of the Society for the year 1954 as required by the provisions of Rule 25.

The Council: The Council of the Society for the year 1954 consisting of the officers and other members elected at the last Annual General Meeting and co-opted thereafter, together with the Editorial Secretary, was constituted as follows:

President

Dr. S. K. Banerjee,

Vice-Presidents

Prof. N. R. Sen, Mr. S. Gupta,
Prof. V. V. Narlikar, Dr. B. B. Sen,
Prof. B. R. Seth,

Treasurer

Mr. S. C. Ghosh,

Secretary

Mr. U. R. Burman,

Editorial Secretary

Mr. P. K. Ghosh,

Other Members of the Council

Prof Ram Behari,	Dr B. N. Prasad,
Dr S Ghosh,	Mr. B. C Chatterjee,
Dr. M Ray,	Dr. S. C. Mitra,
Dr S K. Chakrabarty,	Dr. H M Sengupta.
Dr. A. C. Choudhur ,	Dr. M. Dutta,
Mr. N N. Ghosh,	Dr. R. N. Sen,

Representative of the Government of India

Dr. S S. Bhatnagar,

General: The various activities of the Society have been carried on throughout the year in much the same form as in the past few years. The Council has the pleasure to report that on an invitation from the Society, Prof. J. B. S. Haldane, F.R.S, sometimes

Visiting Professor at the Indian Statistical Institute, delivered a lecture on "Some Mathematical Problems in Biology" at a meeting of the Society in September, 1954. The Council conveys its grateful thanks to this distinguished scientist for his very interesting lecture.

Membership: The Council regrets to report the loss of one of the Society's very old Life members in the death of Prof C. E. Cullis, formerly Hardinge Professor of Higher Mathematics in the University of Calcutta. The late Professor Cullis maintained a deep interest in the affairs of the Society upto the last stage of his life, and made a magnificent gift to the Society in his will, to which a reference will be made later in the course of this report. The Council also mourns the sudden death of Dr. S. S. Bhatnagar*, F.R.S., Secretary to the Ministry of Natural Resources and Scientific Research Govt. of India, and a representative of the Government of India in the Council of the Society during the last four years. The death of Dr. Bhatnagar removes an outstanding figure from India's scientific talents, and is a great blow to the continued development of Natural Resources and Scientific Research in India.

The Council desires to report that 10 new names have been added to the Society's list of ordinary members during the year under review.

Meetings during 1954: The Council held 6 meetings during the year and there were 8 ordinary general meetings devoted to the reading of original papers communicated to the Society for publication in its Bulletin.

Publications: During the year under review the Society has published *four* numbers of the Bulletin, namely Vol. 45, Nos. 3,4 and Vol. 46 Nos. 1,2. The Council takes the opportunity to record here the Society's indebtedness to the authorities of the Calcutta University for printing the Bulletin free of charge and to the Officers and members of the staff of the University Press for their valued services.

Exchange of Publications: The transmission of the Society's publications to various countries in the world has been carried on regularly during the year and some new exchange relations have also been established.

Library: It is gratifying to note that there has been no break during the year under review in the run of periodicals which the Society has been subscribing for the last few years, out of grants received from the Central and State Governments. These grants have also enabled the Society to add to its library a collection of some recent standard books on various branches of Pure and Applied Mathematics.

Finance: Prof. C.E. Cullis, a reference to whose death has already been made in this report, in his will bequeathed to the Society a sum of £500 without any condition other than the expression of a wish that the amount be utilised in meeting the Society's publication expenses in such a manner as the Society may think fit. The Council accordingly resolved to invest the legacy in 3 percent G.P. Notes (redeemable in 1986)

*This melancholy event took place on January 1, 1955 and is included in the present report, though strictly speaking it falls outside the year under review.

under the name "C. E. Cullis Fund" and to spend the annually accrued interest in publications of special nature.

The annual accounts of the Society for the year under review have been presented to the Council in the standardised form by the auditors Dr. N. L Ghosh and Mr. M. C. Chaki. The Council gratefully acknowledges its indebtedness to them for their honorary services. The Society received the following grants during the year under review.

(*i*) Government of West Bengal ... Rs. 1,000/-
(*ii*) National Institute of Sciences of India Rs. 500/-

and the Council takes this opportunity to offer its grateful thanks to the Government of West Bengal and the National Institute of Sciences for these grants.

CALCUTTA MATHEMATICAL SOCIETY

Receipts and Disbursements Accounts of the Calcutta Mathematical Society for the Year Ending 31st December, 1954.

Receipts

		Rs. As. P.	Rs. As. P.
1. Opening Balance			
(a) With Secretary			
(i) In cash ...	2 0 3		
(ii) In stamps ...	3 0 0		
		5 0 3	
(b) Balance at Banks			
(i) Imperial Bank of India	3,625 10 0		
(ii) Do (K. K. G. P Fund)	695 1 8		
(iii) United Bank of India	888 1 3		
Do (in suspense)	88 12 0		
(iv) Postal Savings Bank	936 13 0		
		6,234 5 11	
(c) G. P. Notes (General Fund) (Face value Rs. 6,000) ...		5,663 11 6	
(d) G. P. Notes (K. K. G. P. Fund) (Face value Rs. 2,000) ...		1,937 7 9	
			13,840 9 5
2. Membership subscription ...	858 4 0		
3. Admission fees ...	80 0 0		
4. Sale proceeds ...	3,120 1 0		
Do (in suspense) ...	36 0 0		
		4,094 5 0	
5. Grants			
(i) Govt. of West Bengal ...	1,000 0 0		
(ii) Nat. Inst. of Sciences ...	500 0 0		
		1,500 0 0	
6. C. E. Cullis Fund (£ 500)		6,654 4 0	
7 Interest			
(a) G. P. Notes (General Fund)	180 0 0		
(b) Do (K. K. G. P. Fund)	60 0 0		
(c) Postal Savings Bank ...	18 7 0		
		258 7 0	
			12,507 0 0
		Total ... Rs.	26,347 9 5

Disbursements

		Rs. As. P.	Rs. As. P.
1. Establishment ...		933 4 0	
2. Meetings ...		118 15 3	
3. Books & Journals (including binding) ...		2,740 6 6	
5. Bulletins			
(a) Compositor's Salary	955 0 0		
(b) Papers, Blocks, Types etc.	666 5 6		
(c) Postage	328 7 6		
		1,949 13 0	
5. Printing & Stationery		112 12 6	
6. Postage (general) ...		147 0 0	
7. Bank charges		19 3 0	
8. Miscellaneous (including conveyance charges)		51 8 9	
			6,072 15 0
Closing Balance			
(a) With Secretary			
(i) In cash	0 0 9		
(ii) In stamps	0 0 6		
		0 1 3	
(b) Balance at Banks			
(i) Imperial Bank of India (Gen. & C. E. Cullis Funds)	9,889 3 3		
(ii) Do (K. K. G. P. Fund)	754 9 8		
(iii) United Bank of India	968 5 0		
Do (in suspense)	36 0 0		
(iv) Postal Savings Bank	1,025 4 0		
		12,673 5 11	
(c) G. P. Notes (General Fund) (Face value Rs. 6,000) ...		5,663 11 6	
(d) G. P Notes (K. K. G. P. Fund) (Face value Rs. 2,000) ...		1,937 7 9	
			20,274 10 5
		Total ... Rs.	26,347 9 5

To

The Members of the Calcutta Mathematical Society,

We have examined the above Balance Sheet with the Books and Vouchers relating thereto and certify it to be correctly drawn up therefrom and in accordance with the information and explanations given to us.

N. L. GHOSH,
M. C. CHAKI.
Auditors.

BOUNDARY LAYER IN A CONVERGING NOZZLE WITH A SPHERICAL SURFACE.

By

Lakshmi Sanyal, *Calcutta,*

(*Communicated by Dr. S. Ghosh—Received October 5, 1953*)

In a recent paper Taylor (1950) has discussed the problem of a swirling jet of liquid which enters tangentially a converging conical chamber and goes out through an outlet considerably smaller than the opening through which the liquid enters the chamber. Neglecting the longitudinal and axial components of the velocity outside the boundary layer in comparison with the swirl velocity, he obtains an approximate solution of the boundary layer equations by the Pohlhausen (1921) method. He finds that the retarded liquid within the boundary layer can not maintain its motion in a circular path but is forced towards the vertex of the cone, so that the liquid flows out of the conical chamber through the boundary layer. In this paper, the procedure of Taylor is applied to the problem of swirling liquid through a converging chamber whose surface is spherical. A comparison with Taylor's result shows that when the radii of the inlet and the outlet and the height of the chamber are given the thickness of the boundary layer at the outlet in a spherical chamber is slightly greater than that in a conical one.

1. To obtain the boundary layer equations at the inner surface of the sphere we use polar coordinates R, θ, φ; with the centre of the sphere as pole and the axis of the converging spherical chamber as the polar axis. If u, v, w be the components of velocity in the directions R, θ, φ; p the pressure, ϱ the density and ν the kinematic coefficient of viscosity, the relevant equations of steady motion symmetrical about an axis are (Goldstein, 1950, p. 105)

$$u\frac{\partial v}{\partial R}+\frac{v}{R}\frac{\partial v}{\partial \theta}+\frac{uv}{R}-\frac{w^2\cot\theta}{R}=-\frac{1}{\varrho R}\frac{\partial p}{\partial \theta}+\nu\left(\nabla^2 v+\frac{2}{R^2}\frac{\partial u}{\partial \theta}-\frac{w}{R^2\sin^2\theta}\right) \tag{1}$$

$$u\frac{\partial w}{\partial R}+\frac{v}{R}\frac{\partial w}{\partial \theta}+\frac{wu}{R}+\frac{v\,w\cot\theta}{R}=\nu\left(\nabla^2 w-\frac{w}{R^2\sin^2\theta}\right) \tag{2}$$

where
$$\nabla^2\equiv\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial}{\partial R}\right)+\frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)$$

and the equation continuity is

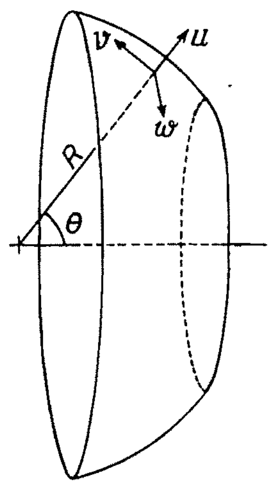
$$\frac{\partial u}{\partial R}+\frac{2u}{R}+\frac{1}{R}\frac{\partial v}{\partial\theta}+\frac{v}{R}\cot\theta=0 \tag{3}$$

If $2\pi\Omega$ be the circulation in the swirling motion, then

$$w=\frac{\Omega}{R\sin\theta}$$

outside the boundary layer and therefore the pressure outside is given by

$$\frac{p}{\varrho} = -\frac{1}{2}\frac{\Omega^2}{R^2 \sin^2\theta} + \text{constant}$$



Assuming the thickness of the boundary layer to be small compared with R, we can neglect the terms uv/R, $v/R^2\sin^2\theta$, $\frac{2}{R^2}\frac{\partial u}{\partial\theta}$ in (1) and replace $\nabla^2 v$ by its largest term $\frac{\partial^2 v}{\partial R^2}$. In (2) we can neglect the term uw/R on the left hand side and replace the right hand side by its largest term $\nu\frac{\partial^2 w}{\partial R^2}$. In the equation of continuity we can neglect the term $2u/R$. Therefore the approximate equations of motion and the approximate equation of continuity are

$$u\frac{\partial v}{\partial R} + \frac{v}{R}\frac{\partial v}{\partial\theta} - \frac{w^2\cot\theta}{R} + \frac{\Omega^2\cos\theta}{R^3\sin^3\theta} = \nu\frac{\partial^2 v}{\partial R^2} \tag{4}$$

$$u\frac{\partial w}{\partial R} + \frac{v}{R}\frac{\partial w}{\partial\theta} + \frac{vw\cot\theta}{R} = \nu\frac{\partial^2 w}{\partial R^2} \tag{5}$$

and

$$\frac{\partial u}{\partial R} + \frac{1}{R}\frac{\partial v}{\partial\theta} + \frac{v}{R}\cot\theta = 0 \tag{6}$$

2. We neglect the variation of R in the boundary layer and put $R = a$ in the equations. If δ be the thickness of the boundary layer, we integrate (4) and (5) with respect to R from $R = a-\delta$ to $R = a$. From (4) we get

$$\int_{a-\delta}^{a} u\frac{\partial v}{\partial R}\,dR + \int_{a-\delta}^{a}\frac{v}{a}\frac{\partial v}{\partial\theta}dR - \int_{a-\delta}^{a}\frac{w^2\cot\theta}{a}\,dR + \Omega^2\int_{a-\delta}^{a}\frac{\cos\theta}{a^3\sin^3\theta}\,dR = \nu\left[\frac{\partial v}{\partial R}\right]_{R=a}$$

Now
$$\int_{a-\delta}^{a} u\,\frac{\partial v}{\partial R}\,dR = uv\Big|_{a-\delta}^{a} - \int_{a-\delta}^{a} v\,\frac{\partial u}{\partial R}\,dR = \int_{a-\delta}^{a}\left[\frac{v}{a}\frac{\partial v}{\partial \theta}+\frac{v^2}{a}\cot\theta\right]dR$$

since the first term vanishes. Hence

$$2\int_{a-\delta}^{a}\frac{v}{a}\frac{\partial v}{\partial \theta}dR+\int_{a-\delta}^{a}\frac{v^2-w^2}{a}\cot\theta\,dR+\frac{\Omega^2\cos\theta}{a^3\sin^3\theta}\int_{a-\delta}^{a}dR = \nu\left[\frac{\partial v}{\partial R}\right]_{R=a} \tag{7}$$

Equation (5) gives

$$\int_{a-\delta}^{a} u\,\frac{\partial w}{\partial R}\,dR+\int_{a-\delta}^{a}\frac{v}{a}\,\frac{\partial w}{\partial \theta}\,dR+\int_{a-\delta}^{a}\frac{vw\cot\theta}{a}\,dR = \nu\left[\frac{\partial w}{\partial R}\right]_{R=a}$$

Now

$$\int_{a-\delta}^{a} u\,\frac{\partial w}{\partial R}\,dR = uw\Big|_{a-\delta}^{a} - \int_{a-\delta}^{a} w\,\frac{\partial u}{\partial R}\,dR$$

$$= -(uw)_{a-\delta}+\int_{a-\delta}^{a}\left[\frac{w}{a}\frac{\partial v}{\partial \theta}+\frac{vw\cot\theta}{a}\right]dR$$

$$= -\frac{\Omega}{a\sin\theta}(u)_{a-\delta}+\int_{a-\delta}^{a}\left[\frac{w}{a}\frac{\partial v}{\partial \theta}+\frac{vw\cot\theta}{a}\right]dR.$$

But
$$(u)_{a-\delta} = -\int_{a-\delta}^{a}\frac{\partial u}{\partial R}\,dR = \int_{a-\delta}^{a}\left[\frac{1}{a}\frac{\partial v}{\partial \theta}+\frac{v}{a}\cot\theta\right]dR.$$

Therefore

$$\int_{a-\delta}^{a} u\,\frac{\partial w}{\partial R}\,dR = -\frac{\Omega}{a\sin\theta}\int_{a-\delta}^{a}\left[\frac{1}{a}\frac{\partial v}{\partial \theta}+\frac{v}{a}\cot\theta\right]dR+\int_{a-\delta}^{a}\left[\frac{w}{a}\frac{\partial v}{\partial \theta}+\frac{vw\cot\theta}{a}\right]dR.$$

Hence

$$-\frac{\Omega}{a\sin\theta}\int_{a-\delta}^{a}\left[\frac{1}{a}\frac{\partial v}{\partial \theta}+\frac{v}{a}\cot\theta\right]dR+\int_{a-\delta}^{a}\frac{1}{a}\,\frac{\partial(vw)}{\partial \theta}\,dR+2\int_{a-\delta}^{a}\frac{vw\cot\theta}{a}\,dR = \nu\left[\frac{\partial w}{\partial R}\right]_{R=a} \tag{8}$$

These are the momentum integrals of the boundary layer.

3. Following Taylor, let us assume

$$v = \frac{\Omega F(\theta)f(\eta)}{a\sin\theta} = \frac{\Omega F}{a\sin\theta}(\eta-2\eta^2+\eta^3)$$

$$w = \frac{\Omega\varphi(\eta)}{a\sin\theta} = \frac{\Omega}{a\sin\theta}(2\eta-\eta^2).$$

where
$$\eta = (a-R)/\delta.$$

These give $v = w = 0$ when $R = a$ and

$$v = 0,\quad w = \frac{\Omega}{a\sin\theta};\quad \frac{\partial v}{\partial R} = 0,\quad \frac{\partial w}{\partial R} = 0 \text{ when } R = a-\delta.$$

We have
$$\left(\frac{\partial v}{\partial\theta}\right)_R = \frac{\Omega f}{a}\left[\frac{1}{\sin\theta}\frac{dF}{d\theta} - \frac{F\cos\theta}{\sin^2\theta}\right] + \frac{\Omega F}{a\sin\theta}\left(\frac{\partial f}{\partial\eta}\right)\left(\frac{\partial\eta}{\partial\theta}\right)_R$$

$$= \frac{\Omega}{a\sin\theta}\left[\left(\frac{dF}{d\theta} - F\cot\theta\right)f - F\left(\frac{\partial f}{\partial\eta}\right)\frac{\eta}{\delta}\frac{d\delta}{d\theta}\right]$$

$$vw = \frac{\Omega^2 F f\varphi}{a^2\sin^2\theta}$$

$$\left.\frac{\partial(vw)}{\partial\theta}\right]_R = \frac{\Omega^2}{a^2\sin^2\theta}\left[\left(\frac{dF}{d\theta} - 2F\cot\theta\right)f\varphi - F\frac{\partial(f\varphi)}{\partial\eta}\frac{\eta}{\delta}\frac{d\delta}{d\theta}\right]$$

We substitute these expressions in (7) and (8) and change the variable of integration from dR to $-\delta d\eta$.

Then since $\displaystyle\int_0^1 \eta f\frac{\partial f}{\partial\eta}d\eta = -\tfrac{1}{2}\int_0^1 f^2 d\eta$ and $\displaystyle\int_0^1 \eta\frac{\partial(f\varphi)}{\partial\eta}d\eta = -\int_0^1 f\varphi\, d\eta$

and f vanishes at both limits of integration, the equations become

$$\frac{\delta^2}{a^2}\left[\left(\frac{dF^2}{d\theta} + \frac{F^2}{2\delta^2}\frac{d\delta^2}{d\theta} - F^2\cot\theta\right]\int_0^1 f^2\, d\eta + \cot\theta\left(1 - \int_0^1 \varphi^2\, d\eta\right)\right] = -\frac{\nu F\sin\theta}{\Omega}\left(\frac{\partial f}{\partial\eta}\right)_{\eta=0} \quad (9)$$

$$\frac{1}{2}\frac{\delta^2}{a^2}\left(\frac{dF^2}{d\theta} + \frac{F^2}{\delta^2}\frac{d\delta^2}{d\theta}\right)\left(-\int_0^1 f d\eta + \int_0^1 f\varphi\, d\eta\right) = -\frac{\nu F\sin\theta}{\Omega}\left(\frac{\partial\varphi}{\partial\eta}\right)_{\eta=0}. \quad (10)$$

Introducing the non-dimensional variable

$$\delta_1 = \delta(\Omega/\nu)^{\frac{1}{2}}/a, \quad (11)$$

the two equations reduce to

$$\left(\frac{dF^2}{d\theta} + \frac{F^2}{2\delta_1^2}\frac{d\delta_1^2}{d\theta} - F^2\cot\theta\right)\int_0^1 f^2\, d\eta + \cot\theta\left(1 - \int_0^1 \varphi^2\, d\eta\right) = -\frac{F\sin\theta}{\delta_1^2}\left(\frac{\partial f}{\partial\eta}\right)_{\eta=0} \quad (12)$$

$$\frac{1}{2}\left(\frac{dF^2}{d\theta} + \frac{F^2}{\delta_1^2}\frac{d\delta_1^2}{d\theta}\right)\left(-\int_0^1 f d\eta + \int_0^1 f\varphi\, d\eta\right) = -\frac{F\sin\theta}{\delta_1^2}\left(\frac{\partial\varphi}{\partial\eta}\right)_{\eta=0} \quad (13)$$

With our values of f and φ, we get

$$\int_0^1 f^2 d\eta = \frac{1}{105},\quad \int_0^1 \varphi^2 d\eta = \frac{8}{15},\quad \int_0^1 f d\eta = \frac{1}{12},\quad \int_0^1 f\varphi d\eta = \frac{1}{20}.$$

Using these integrals and putting $\cos\theta = \mu$, the boundary layer equations can be written as

$$\frac{dF^2}{d\mu} = 98\,\frac{\mu}{1-\mu^2} - \frac{2\mu F^2}{1-\mu^2} + 330\,\frac{F^2}{F\delta_1^2} \tag{14}$$

$$\frac{d(F\delta_1^2)}{d\mu} = -49\,\frac{\mu}{1-\mu^2}\,\frac{F\delta_1^2}{F^2} + \frac{\mu}{1-\mu^2}\,F\delta_1^2 - 285. \tag{15}$$

4. These equations are now solved numerically, using F^2 and $F\delta_1^2$ as dependent variables. The calculation starts with $F^2 = F\delta_1^2 = 0$ when $\mu = \cdot866(\theta = 30^\circ)$ and proceeds at intervals of ·002 to $\mu = \cdot992$ and then at intervals of ·001 to $\mu = \cdot999(\theta = 3^\circ)$ The results of the calcultion are given in Table I.

The angle χ which the direction of flow on the surface of the sphere makes with the meridian section through the point is given by

$$\cot\chi = \left(\frac{\partial v}{\partial\eta}\bigg/\frac{\partial w}{\partial\eta}\right) = \tfrac{1}{2}F.$$

5. For comparison with Taylor's result we take $\mu = \cdot999(\theta = 3^\circ)$ at the outlet and $\mu = \cdot866(\theta = 30^\circ)$ at the inlet. Therefore the radii of the inlet and the outlet are $R_3 = a/2$ and $R_2 = a/20 = R_3/10$ where a is the radius of the sphere. The value of δ_1 at the outlet is found from the Table to be $\delta_1 = 1{\cdot}049$. This gives

$$\delta/R_2 = 20{\cdot}98\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \geq 20{\cdot}98(\nu/UR_2)^{\frac{1}{2}}$$

where $P = \frac{1}{2}\varrho U^2$ is the pressure, which drives the liquid into the chamber.

For a conical chamber of the same height and radii of the openings R_2, R_3 the semi-vertical angle of the cone is $73^\circ{\cdot}5$.

Taylor (1950, p. 139) finds in this case

$$\delta/R_2 = 20(\sin 73^\circ{\cdot}5)^{-\frac{1}{2}}\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}}$$

$$= 20\,{\cdot}425\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}}.$$

This is slightly less than the corresponding quantity for the spherical chamber.

If water ($\nu = 0{\cdot}01$) is forced through the chamber at 10 atmospheres, $U = 45$ metres per sec and $R_2 = 1$ mm.,

$$\delta/R_2 \geq 20{\cdot}98\left\{\frac{0{\cdot}01}{0{\cdot}1(4{\cdot}5\times 10^3)}\right\}^{\frac{1}{2}}$$

$$= \cdot09890.$$

In this case the boundary layer extends inwards to about one-tenth of the radius of the outlet.

TABLE I

μ	·866	·868	·872	·876	·880	884	·890	·896	902
F	0	·3639	·6294	·8261	·9954	1·1415	1·3405	1·5252	1·7013
δ_1	0	·6031	·8774	·9819	1·0513	1·1106	1·1794	1·2317	1·2726
χ	90	79°41′	72°32′	67°34′	63°33′	60°17′	56°10′	52°40′	49°37′
μ	·908	·914	·920	·926	·932	·938	·940	·942	·944
F	1·8728	2 0422	2·2117	2·3831	2·5581	2 7387	2 8005	2·8632	2·9269
δ_1	1·3047	1·3298	1·3489	1 3629	1·3720	1·3767	1·3773	1 3774	1·3770
χ	46°53′	44°24′	42°0′	40°0′	38°3′	36°9′	35°32′	34°56′	34°21′
μ	·948	·954	·960	·966	·972	·978	·984	·990	·992
F	3·0578	3·2643	3·4865	3 7294	4·0002	4 3105	4·6805	5·1516	5·3462
δ_1	1·3747	1·3676	1·3556	1·3383	1·3147	1·2833	1·2415	1·1887	1·1582
χ	33°12′	31°30′	29°51′	28°13′	26°34′	24°54′	23°9′	21°14′	20°31′
μ	·994	·996	·998	999					
F	5·5312	5·7616	6 0599	6·2510					
δ_1	1·1387	1·1112	1·0737	1·0493					
χ	10°53′	19°9′	18°16′	17°45′					

In conclusion I wish to express my thanks to Dr. S. Ghosh for helpful suggestions and guidance.

References.

Goldstein, S. (1950), *Modern Developments in Fluid Dynamics* (Oxford), 105.
Pohlhausen, K. (1921), *Zeits. f. angew. Math und Mech.*, **1**, 257.
Taylor, G. I. (1950), *Quart. J. Mech and App. Math.*, **3**, 129.

NUMERICAL SOLUTIONS TO FIND OUT SPECTRUM FUNCTION OF ISOTROPIC TURBULENCE WITH A FOURTH POWER LAW FITTING AT SMALL EDDY NUMBERS.

By

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(*Communicated by Prof. N. R. Sen—Received October 8, 1953*)

1. **Introduction.** In the current theory of isotropic turbulence a fairly complete representation of the decay phenomena is possible through a turbulence spectrum function $F(k, t)$ for which a particular integro-differential equation has been proposed by Heisenberg. It was shown by Heisenberg that the equation admits a solution of the form,

$$f(kt^{\frac{1}{2}})/t^{\frac{1}{2}} \qquad (1)$$

which can represent the similarily properties of the spectrum. This solution does not yield the asymptotic form $F \sim Ck^4 (k \to 0)$ due to Lin and Batchelor. It has recently been shown that Heisenberg's integro-differential equation, at the stage when the high-frequency part of the spectrum plays an insignificant role, has the more general solution of the form

$$F(k, t) \sim \text{Const } t^{3c-2} f(kt^c) \qquad (2)$$

where c is an arbitrary constant. $c = \frac{1}{2}$ gives the Heisenberg's spectrum, and $c = \frac{2}{7}$ gives the above fourth power law. It has been suggested that at the early stage a variety of spectra satisfying similarity law may be possible depending on the method of production of turbulence.

As the spectrum function (2) has been obtained in a definite form it lends itself to numerical calculation. Numerical solutions of the form (1) (i.e. of (2) for $c = \frac{1}{2}$) has been tabulated by Chandrasekhar (1950). It appears advisable to tabulate in the same manner the solution (2) for the particular case $c = \frac{2}{7}$, which gives the fourth power law. Two such numerical solutions have been given in this paper.

2. When the spectrum function is assumed to be of the form (2) the function f is given by the equation

$$(2-3c)\int_0^x f(x')dx' - cxf(x) = 2\int_x^\infty \sqrt{\{f(x')/x'^3\}}dx' \times \int_0^x x'^2 f(x')dx' \qquad (3)$$

as was shown by N. R. Sen (1951). Introducing the auxiliary functions

$$g = x^3 f(x) \text{ and } y(x) = \int_0^x x'^2 f(x')dx' \qquad (4)$$

in the manner of Chandrasekhar, we can derive from (3) the following equation

$$cg^{3/2}g'' + (4+g')\,y + 2g^{\frac{1}{2}}(2-cg') - 4g = 0 \qquad (5)$$

where g' means dg/dy and c is the parameter introduced by Sen and which may have values $<\frac{2}{3}$. The differential equation for $c = \frac{2}{7}$, *viz.*,

$$2g^{3/2}g'' + 7y(4+g') + 4g^{\frac{1}{2}}(7-g') - 28g = 0 \tag{6}$$

is of non-linear type. The solution should be found subject to the boundary condition,

as $$x \to 0,\ y \to 0,\ g \to 0.$$

We assume, from a study of the nature of $g-y$ curves given by Chandrasekhar, that such a curve will have a single maximum at a point $y = y_0$ (say), should vanish at the origin, and should asymptotically approach the y-axis at the final stage.

It can be shown that the equation (6) has in the neighbourhood of $y = 0$ a solution

$$g \approx 7y \quad (y \to 0). \tag{7}$$

further at a maximum of y we should have

$$g' = 0 \tag{8}$$

and $$g - g^{\frac{1}{2}} - y < 0 \tag{9}$$

For numerical solution we proceed thus. Choose a value y_0 of y; on the ordinate at y_0 apoint is taken fixing g, say g_0. From this point (y_0, g_0) a solution is started which satisfies (8) and (9). It is possible to conclude in most cases after only a few steps that the particular curve will not pass through the origin. On account of (9) only a limited section of the ordinate near the y-axis is concerned here. In this manner, it is possible to find by trial and error, the right point on the ordinate at y_0 through which the solution of (6) satisfying (8) and (9), will pass through the origin. Equation (7) gives som guidance in this matter.

The solution continued on the increasing side of y is found to approach monotonically the y-axis. In this manner a $(g-y)$ table and the corresponding curve may be constructed. Next we have to solve the equation

$$dx/dy = x/g \tag{10}$$

under the initial conditions $x \to 0$, $y \to 0$, $g \to 0$.

3. To solve (10), we take from a $(g-y)$ table a few sets of values of y and g in the close neighbourhood of $y = 0$ and fit a curve of the form

$$g = 7y + a_1 y^{3/2} + a_1 y^2 + a_3 y^{5/2} \tag{11}$$

through these points; a_1, a_2, a_3 are calculated out. Accepting g to have such a fitting in the part of the curve near the origin, we straightway substitute it in (10) and integrate so that we get (near $y = 0$)

$$\log \frac{x}{x_0} = \tfrac{1}{7}\log y - \frac{2}{49}a_1 y^{\frac{1}{2}} - \frac{1}{49}y\left[a_2 - \frac{a_1^2}{7}\right] - \frac{2}{147}y^{3/2}\left[a_3 - \frac{2a_1a_2}{7} + \frac{a_1^3}{49}\right]$$
$$+ \frac{1}{343}y_2\left[\frac{a_2^2}{2} + a_1a_3 - \frac{3a_1^2a_2}{14} + \frac{a_1^4}{98}\right]$$
$$+ \frac{2}{1715}y^{5/2}\left[2a_2a_3 - \tfrac{3}{7}a_1a_2^2 - 3\frac{a_1^2a_3}{7} + \frac{1}{49}a_1^3a_2 - \frac{a_1^5}{343}\right]$$
$$+ \cdots\cdots\cdots \tag{12}$$

x_0 being a scale constant, we put it equal to 1.

Now we can evaluate x at a point on the segment of the curve fitted (in the proximity of $y = 0$), say at $y = \cdot 001$, and starting with that value of x at $y = \cdot 001$, we numerically integrate (10). This will give us the required solution and the corresponding $(x, f(x))$ curve. Two such $(g-y)$ and the corresponding $(x, f(x))$ solutions found numerically are tabulated below.

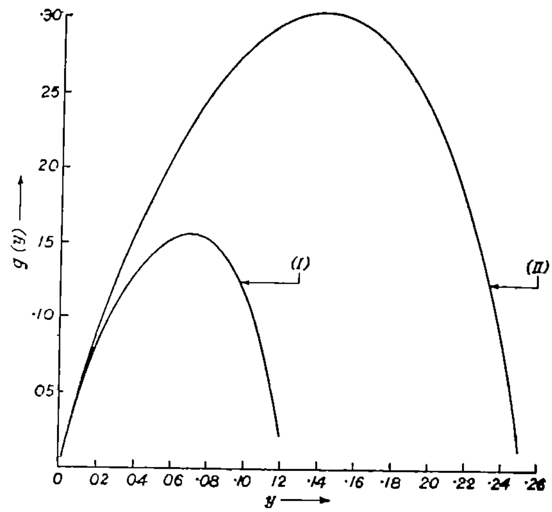

Fig. 1

The solution curves in $g-y$ plane. Curves (I) and (II) represent the following tables (I) and (II) respectively.

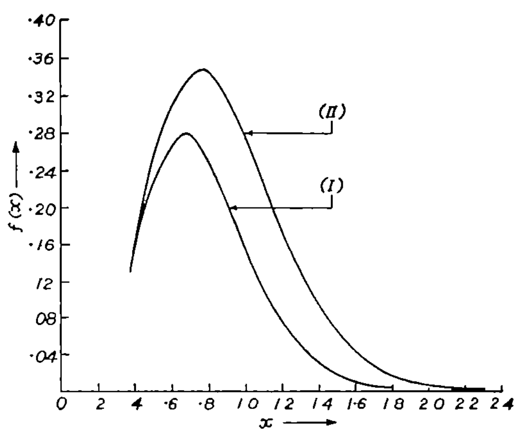

Fig. 2

The decay spectra corresponding to curves in $g-y$ plane with the respective markings (I) and (II).

TABLE I

g_{max}(·15711) at y = ·07...

y	$g(y)$	y	$g(y)$	x	$f(x)$	x	$f(x)$
001	·00686	04	·12956	·3737	·1314	·8006	·2525
·002	·01200	·05	14535	·4166	·1660	·8607	·2279
004	02156	·06	15114	·4708	2067	·9200	·1979
·005	·02608	·07	15711	·4910	·2203	·9810	·1664
·0075	·03676	08	·15307	·5320	·2412	1·0466	·1355
·01	·04674	·09	·14113	·5656	·2592	1·1206	·1003
·02	·08112	·10	11861	·6623	·2792	1·2111	·0668
·03	·10844	·12	·02335	·7360	2720	1 8446	·0037

TABLE II.

g_{max}(·30646) at y = ·14...

y	$g(y)$	y	$g(y)$	x	$f(x)$	x	$f(x)$
·001	·00688	·06	20767	·3736	·1320	·8530	·3346
·002	·01220	·08	·24882	4160	1605	·9310	·3083
·004	02222	·10	·27915	4687	·2158	1·0042	·2757
·005	·02703	12	·29857	4862	·2323	1 0759	·2397
·0075	·03852	·14	30646	·5273	·2628	1·1492	·2019
·01	·04949	·16	·30164	5582	·2845	1·2270	·1633
·02	·08921	·18	28219	·6469	3295	1 3137	·1245
·03	·12404	·20	·24493	·7100	·3452	1·4168	·0861
04	·15513	·22	·18413	·7638	·3481	1·5547	·0490
·05	·18288	·24	·08695	·8104	·3436	1·8078	·0147
		·25	·01220			2·3858	·0009

In conclusion, the author wishes to thank Prof. N. R. Sen for his kind and useful guidance in the preparation of this paper.

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References

Chandrasekhar, S. (1950), *Proc. Roy. Soc*, A, **200**, 20.
Sen, N. R. (1951), *Bull. Cal. Math. Soc*, **43**, 1

THE GENERAL RELATIVITY FIELD OF A RADIATING STAR

By

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(Communicated by the Secretary—Received October 5, 1953)

1. **Introduction**—A non-static solution of the field equations of general relativity, representing the gravitational field of a radiating star, has already been obtained by us (1951). This solution is given by the line-element,

$$ds^2 = -\frac{dr^2}{1-2m/r} - r^2(d\theta^2 + \sin^2\theta d\varphi^2) + \frac{\dot{m}^2}{f^2}\left(1 - \frac{2m}{r}\right)dt^2, \tag{1·1}$$

$$m = m(r, t), \text{ and } m'(1-2m/r) = f(m).$$

In the above and in what follows a dash and a dot indicate differentiations with regard to r and t respectively. The solution is valid within a range $r_0 \leq r \leq r_1$, $t_0 \leq t \leq t_1$ and at (r_0, t_0) as well as at (r_1, t_1), the line-element (1·1) passes off continuously to Schwarzschild's exterior line-element. While studying the possible ways in which a spherical mass of radius r_0 may start emitting radiation at time t_0, it was found that the solution (1·1) was not sufficiently general to explain all such possible ways. A more general solution was desiderated. In what follows we give such a general solution. The notations of the earlier paper (1951) will be assumed throughout. We shall begin by discussing the physical situation which necessitated this generalization.

2. **Beginnings of radiation-emission**—A star of mass M and radius r_0 is supposed to start radiating at time t_0. As the star continues to radiate, the zone of radiation increases in thickness, its outer surface at a later instant t_1 being $r = r_1$. The gravitational field for $r_0 \leq r \leq r_1$ is described in the earlier paper with the help of the line-element (1·1). In the present section we are interested in the situation at the instant t_0. Since, in the outer envelope, the radiation density ϱ is a function of r and t, one has to reckon with different possibilities corresponding to the zero or the non-zero values of ϱ at $r = r_0$, $t = t_0$. One can imagine, for instance, gravitational contraction setting in in a continuous manner. The star contracts, becomes hotter and starts emitting energy. The density ϱ of the emitted energy grows continuously at the stellar boundary. We may then expect that ϱ will be zero at (r_0, t_0). As $4\pi r^2\varrho$ is conserved along the world lines of flow of radiation, ϱ will keep up the value zero on this first wave front which starts with radius r_0 at the instant t_0. At the later instant t_1, this wave front has the radius r_1 and so we expect that for stars which have started emitting radiation in a continuous manner, on the boundary (r_1, t_1) of the radiation zone, we must have

$$\varrho = 0. \tag{2·1}$$

We shall call such stars, the normal stars.

One can also imagine another set of happenings giving rise to $\varrho = \varrho_0 \neq 0$ at (r_0, t_0). In case of such a sudden outburst of radiation, the value of ϱ at the exterior boundary (r_1, t_1) is

$$\varrho_1 = r_0^2 \varrho_0 / r_1^2 \neq 0. \tag{2·2}$$

We shall call such stars, the nova-type stars.

If we now try to represent the field of a normal star by our solution given by (1·1) we shall find that the condition (2·1) means that at the boundary (r_1, t_1), we must have

$$\varrho = 0, \ T_\mu^\nu = 0, \ m' = 0, \ \dot{m} = 0, \ f(M) = 0 \tag{2·3}$$

so that all the successive derivatives of m with regard to r and t will have to vanish over the boundary (r_1, t_1). This is possible in (1·1) only if m is a constant throughout the field $r_0 \leq r \leq r_1$, $t_0 \leq t \leq t_1$. (1·1) then degenerates in Schwarzschild's static solution and so it cannot describe the gravitational field of a radiating normal star.

The boundary conditions for a nova-type star can, however, be satisfied by the solution (1·1). It should, therefore, be regarded as describing the gravitational field of a nova-type star only. Since the radiation-envelope of a normal star cannot be described by the solution (1·1), one expects that a more general solution exists. Such a general solution will now be developed in the next section.

3. The general solution—We take the line-element

$$ds^2 = -e^\lambda dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2) + e^\nu dt^2, \ \lambda = \lambda(r, t), \ \nu = \nu(r, t). \tag{3·1}$$

The differential equations satisfied by the functions λ and ν in order that (3·1) may represent the field of a radiating star, have already been derived in the earlier paper (1951). They are:

$$e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} + \frac{\dot{\lambda}}{r} e^{-\frac{1}{2}(\lambda+\nu)} = 0. \tag{3·2}$$

$$e^{-\lambda}\left(\frac{\lambda' - \nu'}{r} - \frac{2}{r^2}\right) + \frac{2}{r^2} = 0. \tag{3·3}$$

$$e^{-\lambda}\left(\frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\lambda'\nu'}{4} + \frac{\nu' - \lambda'}{2r}\right) + e^{-\nu}\left(\frac{\ddot{\lambda}}{2} + \frac{\dot{\lambda}^2}{4} - \frac{\dot{\lambda}\dot{\nu}}{4}\right) = 0. \tag{3·4}$$

We now try to solve them by putting

$$e^{-\lambda} = 1 - 2m/r, \ m = m(x), \ x = x(r, t). \tag{3·5}$$

(3.2) now leads to

$$e^{-\frac{1}{2}\lambda}\frac{\partial x}{\partial r} + e^{-\frac{1}{2}\nu}\frac{\partial x}{\partial t} = 0,$$

i.e.,

$$\frac{dx}{dr} = 0. \tag{3.6}$$

We can now express $e^{\frac{1}{2}\nu}$ in terms of x:

$$e^{\frac{1}{2}\nu} = -\frac{\dot{x}}{x'}\left(1 - \frac{2m}{r}\right)^{-\frac{1}{2}}. \tag{3.7}$$

On substituting the values of λ and ν from (3.5) and (3.7) we find that the equation (3.3) reduces to

$$\left(\frac{\dot{x}'}{\dot{x}}-\frac{x''}{x'}\right)\left(1-\frac{2m}{r}\right)=\frac{2m}{r^2}. \tag{3.8}$$

The first integral of the above equation is

$$x'(1-2m/r)=f(x). \tag{3.9}$$

As in the earlier paper (1951) it can now be verified that the equation (3.4) is automatically satisfied by virtue of (3.5), (3.7) and (3.9) Hence the final form of the general line-element to describe the gravitational field of a radiating star is

$$ds^2=-\left(1-\frac{2m}{r}\right)^{-1}dr^2-r^2(d\theta^2+\sin^2\theta d\varphi^2)+\frac{\dot{x}^2}{f^2}\left(1-\frac{2m}{r}\right)dt^2 \tag{3.10}$$

with the single relation

$$x'(1-2m/r)=f(x)$$

to connect the three functions $m=m(x)$, $f=f(x)$ and $x=x(r,t)$.

The surviving components of the energy-tensor are the same as in the earlier solution (1943).

$$-T_1^{\ 1}=T_4^{\ 4}=\frac{m'}{4\pi r^2},\quad T_1^{\ 4}=\frac{m'^2}{4\pi \dot{m}r^2},\quad T_4^{\ 1}=\frac{-\dot{m}}{4\pi r^2}.$$

We have only to note that now

$$m'=\frac{dm}{dx}x',\quad \dot{m}=\frac{dm}{dx}\dot{x}.$$

4. The boundary conditions. We can now make our line-element (3.10) continuous with the Schwarzschild's line-element for a constant mass M

$$ds^2=-(1-2M/r)^{-1}dr^2-r^2(d\theta^2+\sin^2\theta d\varphi^2)+(1-2M/r)\,dt^2. \tag{4.1}$$

The boundary radius $r=R(t)$ of the radiation-zone is given by

$$\dot{R}=1-2M/R$$

so that if its value is r_1, at any time t_1, we shall have

$$r_1+2M\log(r-2M)-t_1=\text{a constant}. \tag{4.2}$$

At the boundary (r_1, t_1) we shall find that x is a constant $=X$. Then $m(X)=M$ and $\dot{x}=-f(X)$ will make $g_{\mu\nu}$ continuous at (r_1, t_1) If we wish to describe the field of a normal star for which ϱ vanishes over the boundary, then at (r_1, t_1) we need put

$$\varrho=0,\ i.e.,\ T_\mu^{\ \nu}=0,\ i.e.,\ m'=0,\ i.e.,\ dm/dx=0\text{-}$$

We can choose the function $m=m(x)$ in such a way that dm/dx vanishes when $x=X$. But now x' need not vanish over the boundary (r_1, t_1) and so the line-element (3.10) will not degenerate into Schwarzschild's static solution. Thus this more general solution can be used to describe the field of a normal star.

We may mention here that this general solution, however, does not affect the situation considered by Raychaudhuri (1953) where he wants $\varrho = 0$ at a fixed point. Our solution can describe the situation where ϱ vanishes over a boundary expanding or contracting with the velocity of light, but even this general solution will degenerate to Schwarzschild's exterior solution (4.1) when we wish it to give $\varrho = 0$ continuously at the origin $r = 0$.

Before we close this discussion, a point about the continuity of the derivatives of $g_{\mu\nu}$ at (r_1, t_1) may be noted. We have shown elsewhere (1952) that the continuity of these derivatives of $g_{\mu\nu}$ need not be always assumed. Whether the derivatives of $g_{\mu\nu}$ are continuous at a boundary or not, need not be determined once for all Some physical situations will demand their continuity over a boundary while there may be physical situations for which the derivatives of $g_{\mu\nu}$ should not be continuous over a particular boundary. Here, the field of a normal star requires that $\varrho = 0$ at (r_1, t_1) which is equivalent to ν' continuous at (r_1, t_1). But the field of a nova-type star requires $\varrho \neq 0$ at (r_1, t_1) which makes ν' discontinuous at (r_1, t_1). The solution derived here accommodates both these possibilities.

5. **Summary**. A star of mass M and radius r_0 is supposed to start radiating at time t_0. The zone of radiation extends to $r = r_1$ at a later instant $t = t_1$. Vanishing or otherwise of the density of energy ϱ at the surface of the star at the instant t_0 is discussed and it is found that the solution of the field equations of relativity already obtained (1951) is not sufficiently general to include both the above cases. A more general solution embracing both the above cases is derived.

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References

Raychoudhuri, A (1953), *Z. f. Phys*, **135**, 221
Vaidya, P C (1943), *Curr. Sci.*, **12**, 183.
——————, (1951), *Proc. Indian Acad Sc*, **33**, 261.
——————, (1952), *Curr. Sci.*, **21**, 96.

ON CERTAIN THEOREMS ON OPERATIONAL CALCULUS AND SOME PROPERTIES OF THE GENERALISED k-FUNCTION OF BATEMAN.

By

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(*Received—September 14, 1953*)

Section A

1. In this Section I propose to establish certain theorems on operational Calculus and show their applications.

Theorem I. *Let* $f(p) \doteqdot h(x)$, $p^{\frac{1}{2}}f(p^{\frac{1}{2}}) \doteqdot \dfrac{1}{(\pi x)^{\frac{1}{2}}}\varphi\left(\dfrac{1}{x}\right)$. *Then* $\varphi(p) \doteqdot h(2x^{\frac{1}{2}})$;

provided $h(x)$ *is positive, continuous in* $(0, \alpha)$ *where* α *is arbitrary and* $h(x)/x^2$ *is bounded in* $(0, \infty)$

We know that $2\pi^{\frac{1}{2}}pe^{-xp^{\frac{1}{2}}} \doteqdot xt^{-3/2}e^{-\frac{1}{4}x^2/t}$ (Mclachlan, 1939). (1·1)

Also
$$f(p) = p\int_0^\infty e^{-px}h(x)\,dx.$$

Therefore
$$\frac{1}{(\pi t)^{\frac{1}{2}}}\varphi\left(\frac{1}{t}\right) \doteqdot p^{\frac{1}{2}}f(p^{\frac{1}{2}}) \doteqdot p\int_0^\infty e^{-xp^{\frac{1}{2}}}h(x)dx$$
$$\doteqdot \frac{1}{2\pi^{\frac{1}{2}}}\int_0^\infty 2\pi^{\frac{1}{2}}pe^{-xp^{\frac{1}{2}}}h(x)dx$$
$$= \frac{1}{2\pi^{\frac{1}{2}}}\int_0^\infty xt^{-3/2}e^{-\frac{1}{4}x^2/t}h(x)dx, \text{ by (1·1)}$$
$$= \frac{1}{\pi^{\frac{1}{2}}}t^{-3/2}\int_0^\infty e^{-x/t}h(2x^{\frac{1}{2}})dx \tag{1·2}$$

Hence
$$\varphi(p) \doteqdot h(2x^{\frac{1}{2}}),$$
change in the order of integrations being permissible under the given conditions.

2. **Theorem II.** *Let* $f(p) \doteqdot h(x)$ *and* $h(2x^{\frac{1}{2}}) \doteqdot \varphi(p)$.

Then
$$p^{\frac{1}{2}}f(p^{\frac{1}{2}}) \doteqdot \frac{1}{(\pi x)^{\frac{1}{2}}}\varphi\left(\frac{1}{x}\right).$$

Since
$$\varphi(x) = x\int_0^\infty e^{-xt}h(2t^{\frac{1}{2}})dt.$$

therefore $$\frac{1}{(\pi x)^{\frac{1}{2}}}\varphi\left(\frac{1}{x}\right) = \frac{1}{2\pi^{\frac{1}{2}}}\int_0^\infty t x^{-3/2} e^{-\frac{1}{4}t^2/x} h(t)dt. \tag{2·1}$$

Let us multiply both sides by e^{-px} and integrate between the limits 0 and ∞. We obtain

$$p\int_0^\infty \frac{1}{(\pi x)^{\frac{1}{2}}} e^{-px}\varphi\left(\frac{1}{x}\right)dx = \frac{p}{2\pi^{\frac{1}{2}}}\int_0^\infty e^{-px}x^{-3/2}dx\int_0^\infty t e^{-\frac{1}{4}t^2/x}h(t)dt$$

$$= \frac{p}{2\pi^{\frac{1}{2}}}\int_0^\infty h(t)dt\int_0^\infty t x^{-3/2}e^{-px-\frac{1}{4}t^2/x}dx$$

$$= p\int_0^\infty e^{-p^{\frac{1}{2}}t}h(t)dt$$

$$= p^{\frac{1}{2}}f(p^{\frac{1}{2}}). \tag{2·2'}$$

Hence $$p^{\frac{1}{2}}f(p^{\frac{1}{2}}) \doteqdot \frac{1}{(\pi x)^{\frac{1}{2}}}\varphi\left(\frac{1}{x}\right),$$

again a change in the order of integrations being permissible.

Examples. (1) $h(x) = 2^{-n}x^n J_n(x)$, $f(p) = \dfrac{\Gamma(n+\frac{1}{2})}{\pi^{\frac{1}{2}}}\dfrac{p}{(1+p^2)^{n+\frac{1}{2}}}$, $n > -\frac{1}{2}$

$$p^{\frac{1}{2}}f(p^{\frac{1}{2}}) = \frac{\Gamma(n+\frac{1}{2})}{\pi^{\frac{1}{2}}}\frac{p}{(1+p)^{n+\frac{1}{2}}} \doteqdot \frac{1}{\pi^{\frac{1}{2}}}e^{-x}x^{n-\frac{1}{2}} = \frac{1}{(\pi x)^{\frac{1}{2}}}\varphi\left(\frac{1}{x}\right).$$

Hence $$\varphi(x) = x^{-n}e^{-1/x}.$$

By theorem I, we get $p^{-n}e^{-1/p} \doteqdot h(2x^{\frac{1}{2}}) = x^{\frac{1}{2}n}J_n(2x^{\frac{1}{2}}), n > -1.$

(2) $h(x) = x^{n-1}e^{-\frac{1}{2}x^2}$, $f(p) = \Gamma(n)pe^{\frac{1}{4}p^2}D_{-n}(p)$, $n > 0$

$$\varphi(p) = 2^{n-1}\Gamma\left(\frac{n+1}{2}\right)\frac{p}{(p+2)^{\frac{1}{2}(n+1)}}.$$

By theorem II, $\dfrac{\pi^{\frac{1}{2}}}{2^{\frac{1}{2}n-1}}\dfrac{\Gamma(n)}{\Gamma(\frac{1}{2}n+\frac{1}{2})}pe^{\frac{1}{2}p}D_{-n}((2p)^{\frac{1}{2}}) \doteqdot \dfrac{x^{\frac{1}{2}n-1}}{(1+x)^{\frac{1}{2}(n+1)}}$

or $$2^{\frac{1}{2}n}\Gamma(\tfrac{1}{2}n)pe^{\frac{1}{2}p}D_{-n}((2p)^{\frac{1}{2}}) \doteqdot \frac{x^{\frac{1}{2}n-1}}{(1+x)^{\frac{1}{2}(n+1)}} \tag{2 1}$$

(3) $h(x) = x^{n-1}e^{-x^2}$, $f(p) = 2^{-\frac{1}{2}n}\Gamma(n)pe^{p^2/8}D_{-n}\left(\dfrac{p}{\sqrt{2}}\right)$

$$p^{\frac{1}{2}}f(p^{\frac{1}{2}}) \doteqdot \frac{2\Gamma(\frac{1}{2}n+\frac{1}{2})}{\pi^{\frac{1}{2}}}\frac{(4x)^{\frac{1}{2}n-1}}{(1+4x)^{\frac{1}{2}(n+1)}} = \frac{1}{(\pi x)^{\frac{1}{2}}}\varphi\left(\frac{1}{x}\right)$$

By theorem I, $$2^{n-1}\,\Gamma(\tfrac{1}{2}n+\tfrac{1}{2})\frac{p}{(p+4)^{\frac{1}{2}(n+1)}} \doteqdot (2x^{\frac{1}{2}})^{n-1}e^{-\frac{1}{2}x} \tag{2·2}$$

(4) $f(p)=p^{2k-\lambda}e^{\frac{1}{2}p^2}W_{-k,\,m}(p^2)$, $h(x)\doteqdot \dfrac{x^{\lambda}}{\Gamma(1+\lambda)}\,{}_2F_2(\tfrac{1}{2}+k+m,\ \tfrac{1}{2}+k-m\,;$

$\tfrac{1}{2}(1+\lambda),\ \tfrac{1}{2}\lambda+1\,,\ -\tfrac{1}{4}x^2)$, $\qquad (R(\lambda)>-1)$

$$p^{\frac{1}{2}}f(p^{\frac{1}{2}}) = p^{k-\frac{1}{2}(\lambda-1)}e^{\frac{1}{2}p}W_{-k,\,m}(p) \doteqdot \frac{x^{\frac{1}{2}(\lambda-1)}}{\Gamma(\frac{1}{2}(1+\lambda))}\,{}_2F_1(\tfrac{1}{2}+k+m,\ \tfrac{1}{2}+k-m\,;\ \tfrac{1}{2}(\lambda+1)\,;\ -x)$$

$$= \frac{1}{(\pi x)^{\frac{1}{2}}}\varphi\left(\frac{1}{x}\right).$$

Hence $$x^{\frac{1}{2}\lambda}\ {}_2F_2(\tfrac{1}{2}+k+m,\ \tfrac{1}{2}+k-m\,;\ \tfrac{1}{2}(\lambda+1),\ \tfrac{1}{2}\lambda+1\,;\ -x)$$

$$\doteqdot \frac{\pi^{\frac{1}{2}}\Gamma(\lambda+1)}{2^{\lambda}\Gamma(\frac{1}{2}\lambda+\frac{1}{2})}\,p^{-\frac{1}{2}\lambda}\,{}_2F_1(\tfrac{1}{2}+k+m,\ \tfrac{1}{2}+k-m\,;\ \tfrac{1}{2}(\lambda+1)\,;\ -1/p)$$

$$\doteqdot \Gamma(1+\tfrac{1}{2}\lambda)p^{-\frac{1}{2}\lambda}\,{}_2F_1(\tfrac{1}{2}+k+m,\ \tfrac{1}{2}+k-m\,,\ \tfrac{1}{2}(\lambda+1)\,;\ -1/p). \tag{2·3}$$

(5) $h(x) = x^{n+1}J_n(x)$, $f(p) = \dfrac{2^{n+1}}{\pi^{\frac{1}{2}}}\,\Gamma(n+3/2)\dfrac{p^2}{(1+p^2)^{n+3/2}}$

$$p^{\frac{1}{2}}f(p^{\frac{1}{2}}) = \frac{2^{n+1}}{\pi^{\frac{1}{2}}}\,\Gamma(n+3/2)\ \frac{p^{3/2}}{(1+p)^{n+3/2}}$$

Also (Mclachlan and Humbert, 1941)

$$x^{m-\frac{1}{2}}e^{\frac{1}{2}x}M_{s+m+\frac{1}{2},\,m}(x) \doteqdot \Gamma(2m+1)\,\frac{(p-1)^s}{p^{s+2m}}.$$

Let $s=\tfrac{1}{2},\ m=\tfrac{1}{2}n.$

Then by theorem I, we get

$$x^{\frac{1}{2}(n+1)}J_n(2x^{\frac{1}{2}}) \doteqdot \frac{\Gamma(n+3/2)}{\Gamma(n+1)}\,p^{-\frac{1}{2}n}e^{-1/(2p)}M_{\frac{1}{2}n+1,\,\frac{1}{2}n}\left(\frac{1}{p}\right) \tag{2·4}$$

(6) $h(x) = J_n(2x^{\frac{1}{2}})\,I_n(2x^{\frac{1}{2}})$, $f(p) = J_n\left(\dfrac{2}{p}\right)$

$p^{\frac{1}{2}}f(p^{\frac{1}{2}}) = p^{\frac{1}{2}}J_n(2/p^{\frac{1}{2}})$, $\dfrac{1}{(\pi x)^{\frac{1}{2}}}\varphi\left(\dfrac{1}{x}\right) = x^{-\frac{1}{2}}J_{\frac{1}{2}(n-1),\,n}(3x^{\frac{1}{2}})$ [Mclachlan and Humbert, 1950].

Hence $$\pi^{\frac{1}{2}}p^{-\frac{1}{2}}J_{\frac{1}{2}(n-1),\,n}(3p^{-\frac{1}{2}}) \doteqdot J_n(2^{3/2}x^{\frac{1}{2}})I_n(2^{3/2}x^{\frac{1}{2}}) \tag{2·5}$$

SECTION B

3. This section is devoted to certain integrals involving ultraspherical polynomials and Bateman's k-Function. We have also deduced certain expansions involving k-Function, which are believed to be new.

The function $P_n^\lambda(x)$ is defined by the relation

$$P_n^\lambda(x) = \frac{(-2)^n\Gamma(n+\lambda)\Gamma(n+2\lambda)}{n!\Gamma(\lambda)\Gamma(2n+2\lambda)}(1-x^2)^{\frac{1}{2}-\lambda}\frac{d^n}{dx^n}\{(1-x^2)^{n+\lambda-\frac{1}{2}}\}. \tag{3·1}$$

It has also been shown by A. Sharma (1951) that

$$\int_0^1 P_n^\lambda(1-2y^2)(1-y^2)^{\lambda-\frac{1}{2}}y^{2r+2\lambda+2\nu}\,dy$$

$$= \frac{(-1)^n\pi^{\frac{1}{2}}\Gamma(n+2\lambda)\Gamma(\nu+r+1)\Gamma(\nu+r+\lambda+\frac{1}{2})}{2^{2\lambda}n!\Gamma(\lambda)\Gamma(\nu+r-n+1)\Gamma(\nu+r+n+2\lambda+1)}, \quad \begin{array}{l}\lambda>-\frac{1}{2},\\ \nu+r+\lambda>-\frac{1}{2}\\ n = \text{a positive integer.}\end{array} \tag{3·2}$$

We also know that

$$e^{\frac{1}{2}x}k^{2l}_{2n}(\tfrac{1}{2}x) = \frac{(-1)^{n-l-1}x^{2l+1}}{\Gamma(2l+2)}\,{}_1F_1(l-n+1\,;\,2l+2\,;\,x), \quad x>0$$

($n \geq l+1$ and $n-l$ is a positive integer).

Therefore

$$\int_0^1 P_n^\lambda(1-2y^2)(1-y^2)^{\lambda-\frac{1}{2}}y^{2\lambda+2\nu}e^{\frac{1}{2}xy^2}k^{2l}_{2s}(\tfrac{1}{2}xy^2)\,dy$$

$$=\frac{(-1)^{s-l-1}x^{2l+1}}{\Gamma(2l+2)}\int_0^1 P_n^\lambda(1-2y^2)(1-y^2)^{\lambda-\frac{1}{2}}\sum_{r=0}^{\infty}\frac{(l-s+1)_r}{r!\,(2l+2)_r}x^r y^{2\lambda+2r+2\nu+4l+2}\,dy$$

$$=\frac{(-1)^{s-l-1}x^{2l+1}}{\Gamma(2l+2)}\sum_{r=0}^{\infty}\frac{(l-s+1)_r}{r!\,(2l+2)_r}x^r\int_0^1 P_n^\lambda(1-2y^2)(1-y^2)^{\lambda-\frac{1}{2}}y^{2\lambda+2r+2\nu+4l+2}\,dy$$

$$=\frac{(-1)^{n+s-l-1}\pi^{\frac{1}{2}}\Gamma(n+2\lambda)}{2^{2\lambda}n!\,\Gamma(\lambda)\Gamma(2l+2)}x^{2l+1}\sum_{r=0}^{\infty}\frac{(l-s+1)_r}{r!\,(2l+2)_r}\frac{\Gamma(r+\nu+2l+2)\Gamma(r+\nu+\lambda+2l+\frac{3}{2})x^r}{\Gamma(r+\nu-n+2l+2)\Gamma(r+n+2\lambda+\nu+2l+2)}$$

$$=\frac{(-1)^{n+s-l-1}\pi^{\frac{1}{2}}\Gamma(n+2\lambda)\Gamma(\nu+2l+2)\Gamma(\lambda+\nu+2l+\frac{3}{2})\,x^{2l+1}}{2^{2\lambda}n!\,\Gamma(\lambda)\Gamma(2l+2)\Gamma(\nu-n+2l+2)\Gamma(n+2\lambda+\nu+2l+2)}\times$$

$${}_3F_3\left(\begin{matrix}l-s+1,\ \nu+2l+2,\ \lambda+\nu+2l+3/2,\\ 2l+2,\ \nu-n+2l+2,\ n+2\lambda+\nu+2l+2\,;\end{matrix}\ x\right);$$

$$\lambda+\nu+2l+3/2>0 \tag{3·3}$$

change in the order of integration and summation being easily justifiable. *

(*i*) Let $n = l+s+\nu+1$ and $2\lambda+2\nu = 1$

Since $l-s+1$ is either zero or a negative integer, we get

$$\int_0^1 P^\lambda_{l+s+1}(1-y^2)(1-y^2)^{\lambda-\frac{1}{2}}y^{2\lambda}e^{\frac{1}{2}xy^2}k^{2l}_{2s}(\tfrac{1}{2}xy^2)dy = 0 \tag{3·4}$$

* A similar result has been obtained by Srivastava independently.

(ii) $\int_0^1 P_{l+s+1}(1-2y^2)ye^{\frac{1}{2}xy^2}k_{2s}^{2l}(\frac{1}{2}xy^2)dy = 0$ (3·5)

Now

$$e^{\frac{1}{2}x}k_{2n}^{2l}(\tfrac{1}{2}x) \doteqdot (1-p)^{n-l-1}/p^{n+l}.$$

Hence

$$e^{\frac{1}{2}x}\sum_{r=0}^{\infty}\frac{(-1)^r t^r}{r!}k_{2(m+r+1)}^{2m}(\tfrac{1}{2}x) \doteqdot \sum_{r=0}^{\infty}\frac{(-1)^r t^r}{r!}\,\frac{(1-p)^r}{p^{2m+r+1}}$$

$$= \frac{1}{p^{2m+1}}e^{t-t/p}$$

$$\doteqdot e^t\left(\frac{x}{t}\right)^{m+\frac{1}{2}}J_{2m+1}(2(xt)^{\frac{1}{2}})$$

Therefore by Lerch's theorem

$$e^{-\frac{1}{2}x}e^t\left(\frac{x}{t}\right)^{m+\frac{1}{2}}J_{2m+1}(2(xt)^{\frac{1}{2}}) = \sum_{r=0}^{\infty}\frac{(-1)^r t^r}{r!}k_{2(m+r+1)}^{2m}(\tfrac{1}{2}x)\,;\ |\,t\,| < 1. \tag{3·6}$$

Dividing both sides by t^{r+1} and integrating round a contour C, we get

$$e^{\frac{1}{2}x}\frac{(-1)^r}{r!}k_{2(m+r+1)}^{2m}(\tfrac{1}{2}x) = \frac{1}{2\pi i}x^{m+\frac{1}{2}}\int_C e^t t^{-m-r-3/2}J_{2m+1}(2(xt)^{\frac{1}{2}})dt. \tag{3·7}$$

Let us put $t = 1/p$ in (3·6) and multiply both sides by $e^{-1/p}p^{-2m-1}$ and interpret. We get

$$\frac{(xy)^{2m+1}}{\{\Gamma(2m+2)\}^2}\,{}_0F_2(2m+2,\,2m+2\,;\,-xy)$$

$$= \sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\,y^{m+\frac{1}{2}+\frac{1}{2}r}J_{2m+1+r}(2y^{\frac{1}{2}})e^{\frac{1}{2}x}k_{2(m+r+1)}^{2m}(\tfrac{1}{2}x). \tag{3·8}$$

Again let us put $t = a(1-p)/p$ and multiply both sides of (3·6) by p^{-2m-1}.

On interpretation the right hand side becomes

$$\sum_{r=0}^{\infty}\frac{(-1)^r a^r}{r!}\,e^{\frac{1}{2}(x+y)}k_{2(m+r+1)}^{2m}(\tfrac{1}{2}x)k_{2(m+r+1)}^{2m}(\tfrac{1}{2}y).$$

The left hand side is

$$\left(\frac{x}{a}\right)^{m+\frac{1}{2}}e^{a(1-p)/p}(p-p^2)^{-m-\frac{1}{2}}J_{2m+1}\{2[ax(1-p)/p]^{\frac{1}{2}}\}.$$

Hence

$$\left(\frac{x}{a}\right)^{m+\frac{1}{2}}e^{a(1-p)/p}(p-p^2)^{-m-\frac{1}{2}}J_{2m+1}\{2[ax(1-p)/p]^{\frac{1}{2}}\}$$

$$\doteqdot \sum_{r=0}^{\infty}\frac{(-1)^r}{r!}a^r e^{\frac{1}{2}(x+y)}k_{2(m+r+1)}^{2m}(\tfrac{1}{2}x)k_{2(m+r+1)}^{2m}(\tfrac{1}{2}y),\ |\,a\,| < 1 \tag{3·9}$$

where $$1/p \doteqdot y$$

Next multiplying both sides of (3·6) by $t^{m+\frac{1}{2}}$ and putting $t = 1/p^2$, we get

$$J_{2m+1}(2x^{\frac{1}{2}}/p) = e^{\frac{1}{2}x}x^{-m-\frac{1}{2}}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\,\frac{e^{-1/p^2}}{p^{2m+2r+1}}\,k^{2m}_{2(m+r+1)}(\tfrac{1}{2}x).$$

On interpretation and writing $y^{\frac{1}{2}}$ for y, we get

$$J_{2m+1}(2(xy)^{\frac{1}{2}})I_{2m+1}(2(xy)^{\frac{1}{2}}) = e^{\frac{1}{2}x}x^{-m-\frac{1}{2}}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}\,\frac{y^{m+\frac{1}{2}+r}}{\Gamma(2m+2r+2)}$$

$$\times\, {}_0F_2(2m+2r+2,\ 2m+2r+3\ ;\ y)k^{2m}_{2(m+r+1)}(\tfrac{1}{2}x) \qquad (3·10)$$

Mitra and Srivastava have shown that

$$\int_0^{\infty} k^{2l}_{2n}(\tfrac{1}{2}x)k^{2l}_{2s}(\tfrac{1}{2}x)\frac{dx}{x^{2l+1}} = 0, \qquad (n \neq s)$$

$$= \frac{2^{2l}\{\Gamma(2l+2)\}^2\Gamma(n-l)}{\Gamma(n+l+1)},\ (n = s). \qquad (3·11)$$

Hence

$$\int_0^{\infty} e^{-\frac{1}{2}x}x^{-(m+\frac{1}{2})}J_{2m+1}(2(xy)^{\frac{1}{2}})I_{2m+1}(2(xy)^{\frac{1}{2}})k^{2m}_{2(m+r+1)}(\tfrac{1}{2}x)dx$$

$$= \frac{(-1)^r 2^{2m}\{\Gamma(2m+2)\}^2}{\Gamma(2m+r+2)\Gamma(2m+2r+2)}\,y^{m+\frac{1}{2}+r}\,{}_0F_2(2m+2r+2,\ 2m+2r+3\ ;\ y) \qquad (3·12)$$

In conclusion I wish to express my indebtedness to Dr. S. C. Mitra for his help and guidance in the preparation of this paper,

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References

Mclachlan, N W., (1939) *Complex Variable and Operational Calculus*, p. 209.
Mclachlan, N. W. and Humbert, P., (1941) *Memorial des Sciences Mathematiques Fascicules*, 100, 50 and 113,
——————— (1950) *Loc. cit.*, *Fascicule* 113, 11.
Mitra and Srivastava, *In the press.*
Sharma, A , (1951) *Bull. Cal. Math. Soc.*, **43**, 61,
Srivastava, H. M., *In the press.*

SINGULARLY LOADED RECTILINEAR PLATES—BENDING BY ISOLATED COUPLES

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(*Communicated by Prof B. R. Seth—Received September 1, 1953*)

Abstract. This is a continuation of our study of the problem of bending of a rectilinear plate under concentrated loads. In this paper we consider the case of an isolated couple applied at a point on the axis of symmetry of the plate The results for the deflection and the shearing forces are obtained in terms of elliptic functions and their numerical values tabulated in a number of cases.

1. Introduction

1.1. In a previous paper we have studied the problem of a simply supported rectilinear plate subjected to concentrated loads (Aggarwala). It was shown there, how, with the help of the vortex analogy, the use of nonconvergent or slowly convergent series could be eliminated and the results obtained in terms of elliptic functions.

In this paper we propose to show that the case of an isolated couple applied at any point inside the plate can be tackled on similar lines. The particular cases when the couple is applied at an angular point or at a point on an axis of symmetry, if any,, have been worked out in detail. The expressions for deflection and shearing forces are obtained in terms of elliptic functions and their numerical values tabulated.

We know that the differential equation satisfied by the deflection w is

$$\nabla_1^4 w = 0 \tag{1.1}$$

except at the point of application of the couple, where $\nabla_1^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

The boundary conditions for a simply supported edge are

$$w = 0\,;\ \nabla_1^2 w = 0. \tag{1.2}$$

For a concentrated couple, (1.1) may be written as

$$\nabla_1^2 w = \frac{-P}{4\pi D}\left[\frac{e^{i\theta}}{\zeta-\zeta_0} - \frac{e^{-i\theta}}{\zeta-\bar{\zeta}_0} + \frac{e^{-i\theta}}{\bar{\zeta}-\bar{\zeta}_0} - \frac{e^{i\theta}}{\bar{\zeta}-\zeta_0}\right] \tag{1.3}$$

where

$$\zeta = \zeta(z) \tag{1.4}$$

is the transformation that transforms the interior of the plate in z-plane into the upper half of the ζ-plane.

$\zeta_0 = \zeta(z_0)$, z_0 being the point of application of the couple.

θ = the angle that the axis of the couple in ζ-plane makes with

ξ-axis, $(\zeta = \xi + i\eta)$,

P = the intensity of the couple in the ζ-plane,

D = the flexural rigidity of the plate.

In the particular case when ζ_0 is purely imaginary, and $\theta=-\frac{1}{2}\pi$, (1·3) reduces to

$$\nabla_1^2 w=\frac{iP}{2\pi D}\left[\frac{\zeta}{\zeta^2-\zeta_0^2}-\frac{\bar{\zeta}}{\bar{\zeta}^2-\bar{\zeta}_0^2}\right] \tag{1.5}$$

and when $\zeta_0=0$, (1.5) becomes

$$\nabla_1^2 w=\frac{iP}{2\pi D}\left[\frac{1}{\zeta}-\frac{1}{\bar{\zeta}}\right]. \tag{1.6}$$

We propose to consider these results for the following boundaries:—
(1) Square. (2) Equilateral triangle. (3) Right-angled isosceles triangle. (4) Isosceles triangle containing an angle of 120°. (5) Right-angled triangle containing an angle of 30°.

In (1.6), it is seen that in order that the couple in the ζ-plane may be finite, the couple in the z-plane must be very large. For small or moderately large values of couple in the z-plane the couple in the ζ-plane and the deformations in the plate are found to be very small (see also Appendix II). The solutions for the couple at an angular point do not therefore, seem to be very useful for practical purposes. But, for our present purposes, they are useful in as much as they help us in finding out the function necessary to fit the boundary conditions in the general case of a couple at a point on the axis of symmetry if any. The case of a couple at an angular point has, accordingly, been dealt with first in each case, and then the general case has been tackled. In the last two cases namely, the isosceles triangle containing an angle of 120° and the right-angled triangle with an angle of 30°, only the case of a couple at an angular point has been dealt with. It may be remarked that the case of the isosceles triangle containing an angle of 120° with a couple a point on its axis of symmetry can be deduced from that of an equilateral triangle by the method of images.

2. Square

2.1. Couple at an angular point. In this case we write (1.4) as (Aggarwala)

$$\zeta=\frac{-1}{p(z)},\ g_2=4,\ g_3=0.$$

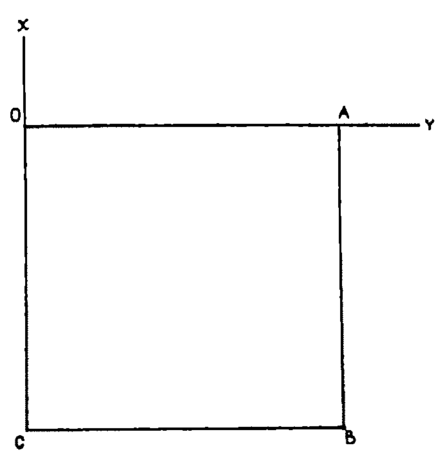


FIG. 1

The angular points O, A, B, C in the z-figure are taken to be 0, ω_1, $-\omega_2$, ω_3 respectively, where ω_1 and ω_3 denote the real and imaginary half periods of the Weierstrassian elliptic function $p(z)$; $\omega_3 = -i\omega_1$ and $\omega_1 + \omega_2 + \omega_3 = 0$ (see fig. 1)

(1 6) now becomes
$$\nabla_1^2 w = \frac{-iP}{2\pi D}[pz - p\bar{z}] \tag{2.1.2}$$

giving
$$-\frac{8\pi i w D}{P} = \bar{z}\zeta z - z\zeta\bar{z} + \varphi, \tag{2.1.3}$$

where φ is harmonic.

It is found that the boundary conditions (1.2) are satisfied if we take

$$\varphi = \frac{-\omega_1}{2\eta_1}(\zeta^2 z - \zeta^2\bar{z}) + \frac{\eta_1}{2\omega_1}(z^2 - \bar{z}^2) + \frac{\omega_1}{2\eta_1}(pz - p\bar{z}) \tag{2.1.4}$$

so that (2 1 3) becomes

$$-\frac{8\pi i w D}{P} = \bar{z}\zeta z - z\zeta\bar{z} - \frac{\omega_1}{2\eta_1}(\zeta^2 z - \zeta^2\bar{z}) + \frac{\omega_1}{2\eta_1}(pz - p\bar{z}) + \frac{\eta_1}{2\omega_1}(z^2 - \bar{z}^2). \tag{2.1.5}$$

For points on OB, (2.1.5) takes the form

$$-\frac{8\pi w D}{P} = 2r\zeta(r, -4, 0) - \frac{\omega_1}{\eta_1}[\zeta^2(r, -4, 0) - p(r, -4, 0] - \frac{\eta_1 r^2}{\omega_1} \tag{2.1.6}$$
$$= 16\pi\alpha, \text{ say,}$$

r denoting the distance from the origin.

Using the values of $p(r, -4, 0)$, $\zeta(r, -4, 0)$ given in

TABLE 1

r	$\omega_1\sqrt{2}/4$	$\omega_1\sqrt{2}/3$	$\omega_1\sqrt{2}/2$	$2\omega_1\sqrt{2}/3$	$3\omega_1\sqrt{2}/4$
$p(r)$	4·611582	2·542461	1	·393320	·216145
$\zeta(r)$	2·164041	1·633730	1·130713	·926895	·880541

we get the values of α given in

TABLE 2

-	$\omega_1\sqrt{2}/4$	$\omega_1\sqrt{2}/3$	$\omega_1\sqrt{2}/2$	$2\omega_1\sqrt{2}/3$	$3\omega_1\sqrt{2}/4$
α	·034845	·0311895	·021769	·011416	·0068248

Shear Forces. From
$$\nabla_1^2 w = \frac{-iP}{2\pi D}(pz - p\bar{z}) \tag{2.1.2 bis}$$

we get
$$Q_x = -D\frac{\partial}{\partial x}\nabla_1^2 w = \frac{iP}{2\pi}(p'z - p'\bar{z})\,;\ Q_y = \frac{-P}{2\pi}(p'z + p'\bar{z}). \tag{2.1.7}$$

For points on OA this gives

$$Q_x = 0\,;\ Q_y = \frac{-P}{\pi}p'x = \frac{2P\alpha}{\pi}, \text{ say.} \tag{2.1.8}$$

the values of σ for a few values of x are given in

TABLE 3

x	$\omega_1/4$	$\omega_1/3$	$\omega_1/2$	$2\omega_1/3$	$3\omega_1/1$
α	28·3359	11.8989	3·41421	1·30191	·816661

For points on the diagonal through the origin, we have,

$$Q_x = \frac{-P\sqrt{2}}{2\pi} p'(r, -4, 0) \,;\; Q_y = \frac{P\sqrt{2}}{2\pi} p'(r, -4, 0) \tag{2.1.9}$$

$$\therefore\; Q_r = Q_x \cos 45^\circ - Q_y \sin 45^\circ = \frac{-P}{\pi} p'(r, -4, 0) \tag{2.1.10}$$

$$= \frac{2P\alpha}{\pi}, \text{ say,}$$

Q_r denoting the shear per unit area across a plane perpendicular to the diameter. The shear in the direction perpendicular to r (which we shall henceforth denote by Q_s) is found to be zero. The values of α for a few values of r are given in

TABLE 4

r	$\omega_1\sqrt{2}/4$	$\omega_1\sqrt{2}/3$	$\omega_1\sqrt{2}/2$	$2\omega_1\sqrt{2}/3$	$3\omega_1\sqrt{2}/4$
α	10·1333	4·35628	1·414214	·678919	·476489

2.2. *Couple at a point on the diagonal through the origin.* In this case we write (1.4) as

$$\zeta = p(z) \,;\; g_2 = 4,\; g_3 = 0. \tag{2.2.1}$$

We take the point of application of the couple on OB, at a distance r_1 from the origin O.

The equation (1.5) in this case becomes

$$\nabla_1^2 w = \frac{iP}{4\pi D}\left[\frac{1}{pz - pz_0} + \frac{1}{pz - p\bar{z}_0} - \frac{1}{p\bar{z} - p\bar{z}_0} - \frac{1}{p\bar{z} - pz_0} \right] \tag{2.2.2}$$

where $z_0 = r_1\, e^{-i\pi/4}$.

Integrating this equation we get

$$\frac{16\pi i w D}{P} p'z_0 = \bar{z} \log \sigma(z + z_0) - \bar{z} \log \sigma(z - z_0)$$

$$- z \log \sigma(\bar{z} + z_0) + z \log \sigma(\bar{z} - z_0)$$

$$- i\bar{z} \log \sigma(z + \bar{z}_0) + i\bar{z} \log \sigma(z - \bar{z}_0)$$

$$+ iz \log \sigma(\bar{z} + \bar{z}_0) - iz \log \sigma(\bar{z} - \bar{z}_0)$$

$$+ \text{a harmonic function.} \tag{2.2.3}$$

Modifying this integral in such a way that we get singularities of the type $r \log r$ at the point z_0 and its images in the lines OA and OC, we write,

$$
\begin{aligned}
\frac{16\pi i w D}{P} p' z_0 = & (\bar{z}+\bar{z}_0)\log\sigma(z+z_0) - (\bar{z}-\bar{z}_0)\log\sigma(z-z_0) \\
& + (\bar{z}+\bar{z}_0)\log\sigma(\bar{z}+\bar{z}_0) - (\bar{z}-\bar{z}_0)\log\sigma(\bar{z}-\bar{z}_0) \\
& - (z+\bar{z}_0)\log\sigma(\bar{z}+z_0) + (z-\bar{z}_0)\log\sigma(\bar{z}-z_0) \\
& - (z+\bar{z}_0)\log\sigma(z+\bar{z}_0) + (z-\bar{z}_0)\log\sigma(z-\bar{z}_0) \\
& + i(z-z_0)\log\sigma(\bar{z}+\bar{z}_0) - i(z-z_0)\log\sigma(\bar{z}-\bar{z}_0) \\
& + i(z+z_0)\log\sigma(z+z_0) - i(z-z_0)\log\sigma(z-z_0) \\
& - i(\bar{z}+z_0)\log\sigma(z+\bar{z}_0) + i(\bar{z}-z_0)\log\sigma(z-\bar{z}_0) \\
& - i(\bar{z}+z_0)\log\sigma(\bar{z}+z_0) + i(\bar{z}-z_0)\log\sigma(\bar{z}-z_0) \\
& + \text{a harmonic function}
\end{aligned}
\tag{2.2.4}
$$

The expression on the right hand side of (2.2.4) excepting the harmonic function vanishes on OA and OC, while on CB and AB its values are

$$
\left.
\begin{aligned}
& -2i\omega_2\left[\log\frac{\sigma(\omega_3+x+z_0)\sigma(-\omega_3+x+\bar{z}_0)}{\sigma(\omega_3+x-z_0)\sigma(-\omega_3+x-z_0)} - 2\eta_3(z_0-\bar{z}_0)\right] + 8\eta_3 x z_0 \\
\text{and}\quad & -2i\omega_2\left[\log\frac{\sigma(\omega_1+iy+z_0)\sigma(\omega_1-iy+\bar{z}_0)}{\sigma(\omega_1+iy-z_0)\sigma(\omega_1-iy-z_0)} - 2\eta_1(z_0+\bar{z}_0)\right] - 8i\eta_1 y z_0
\end{aligned}
\right\}
\tag{2.2.5}
$$

respectively.

In order to find the harmonic function which will vanish on OA and OC and have the values (2.2.5) on CB and AB, we expand the expression in (2.2.5) in Taylor's series and then the required function is found to be

$$
\begin{aligned}
& (2\eta_3/\omega_3)z_0(z^2-\bar{z}^2) - 2i\omega_2[A_1(\zeta^2 z - pz) + A_3(zp'z + 2pz) \\
& + A_5(z\zeta p''' z + 24p^3 z - 16pz) + A_7 z(p^{v} z + 6!\, p^3 z - \tfrac{3}{5}.\, 6!\, pz) \\
& + \text{the complex conjugates}]
\end{aligned}
\tag{2.2.6}
$$

where
$$A_1 = \frac{1}{2!\,\eta_3}(z_0+\bar{z}_0)\,;\; A_3 = \frac{-1}{8!\,\omega_3}(z_0^3+\bar{z}_0^3)\,;\; A_5 = \frac{-1}{5!\,\eta_3}(z_0^5+\bar{z}_0^5) \tag{2.2.7}$$

and in general

$$A_{4n-1} = \frac{-1}{(4n-1)!\,\omega_3}(z_0^{4n-1}+\bar{z}_0^{4n-1})\,;\; A_{4n+1}(n\neq 0) = \frac{-1}{(4n+1)!\,\omega_3}(z_0^{4n+1}+\bar{z}_0^{4n+1}) \tag{2.2.7a}$$

From (2.2.4) and (2.2.6) we have the required expression for w.

Shear Forces: For points on OA, we have

$$Q_x = 0\,;\; Q_y = \frac{-P}{2\pi}\left[\frac{p'x}{(p'x-pz_0)^2} + \frac{p'x}{(px-p\bar{z}_0)^2}\right] \tag{2.2.8}$$

In the particular case when the point of application is the centre, we get,

$$Q_y = \frac{-Pp'x}{\pi}\left[\frac{p^2x-1}{(p^2x+1)^2}\right] = \frac{2P\alpha}{\pi}, \text{ say.} \tag{2.2.9}$$

The values α for a few values of x are given in

TABLE 5

x	$\omega_1/4$	$\omega_1/3$	$\omega_1/2$	$2\omega_1/3$	$3\omega_1/4$
α	·314488	·384051	·353553	·151055	-·068195

Similarly, for points on OB we get

$$Q_r = \frac{P}{\pi}\frac{p'(r,-4,0)[p^2(r,-4,0)+p^2(r_1,-4,0)]}{[p^2(r_1-4,0)-p^2(r_1,-4,0)]^2}$$

$$= \frac{-2P\alpha}{\pi}, \text{ say} \tag{2.2.10}$$

and $\qquad Q_s = 0$

where Q_s and Q_r have the same meaning as in sec. 2·1.

The values of α for the particular case when the couple acts at the centre, are given, for a few values of r, in

TABLE 6

r	$\omega_1\sqrt{2}/4$	$\omega_1\sqrt{2}/3$	$\omega_1\sqrt{2}/2$	$2\omega_1\sqrt{2}/3$	$3\omega_1\sqrt{2}/3$
α	·54934	1.08907	∞	1·08907	·54934

3. Equilateral Triangle

3.1. Couple at an angular point. In this case we write (1.4) as (Love, 1889)

$$\zeta = \frac{2i}{p'z}\ ;\ g_2 = 0;\ g_3 = 4. \tag{3.1.1}$$

The points P, Q, R of the triangle in the z-plane are taken to be $2i\omega_1/\sqrt{3}$, 0, $\omega_1(1+i/\sqrt{3})$, respectively; ω_1 denoting the real half period of $p(z)$ (see fig. 2)

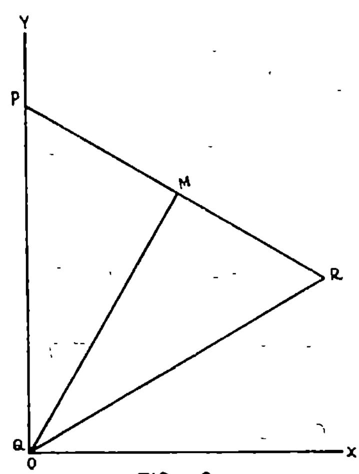


FIG. 2

(1.6) may now be written as

$$\nabla_1^2 w = \frac{P}{4\pi D}(p'z + \overline{p'z}) \tag{3.1.2}$$

giving

$$\frac{16\pi w D}{P} = \bar{z}pz + z\overline{pz} + \varphi \tag{3.1.3}$$

where φ is harmonic.

It is found that on QP and QR

$$\bar{z}pz + z\overline{pz} = 0 \tag{3.1.4a}$$

and on RP

$$\bar{z}pz + z\overline{pz} = -2\omega_1 p(ir) \tag{3.1.4b}$$

where r denotes the distance from C, the point where PR produced meets the axis of x.

The condition $w = 0$ on the boundary is therefore, satisfied if, on QP and QR

$$\varphi = 0 \tag{3.1.5a}$$

and on RP

$$\varphi = 2\omega_1 p(ir). \tag{3.1.5b}$$

To find this function we note that if $G = G(z, t)$ denote Green's function for the triangle, we get

$$\varphi = \int_{2\omega_1/\sqrt{3}}^{4\omega_1/\sqrt{3}} 2\omega_1 p(ir)(\partial G/\partial n)dr \tag{3.1.6}$$

where n denotes the direction of the outward drawn normal.

On PR

$$\frac{\partial}{\partial n} = \cos 60° \frac{\partial}{\partial x} + \sin 60° \frac{\partial}{\partial y}$$

$$= -\varepsilon^2 \frac{\partial}{\partial z} - \varepsilon \frac{\partial}{\partial \bar{z}} \tag{3.1.7}$$

where 1, ε, ε^2 are the cube roots of unity such that

$$\varepsilon = \tfrac{1}{2}(-1 + i\sqrt{3}). \tag{3.1.8}$$

Now

$$4\pi G = \log \frac{\dfrac{2i}{p'z} - \dfrac{2i}{p't}}{\dfrac{2i}{p'z} + \dfrac{2i}{p't}} + \text{complex conjugate}$$

$$= \log \frac{p'z - p't}{p'z + p't} + \text{complex conjugate} \tag{3.1.9}$$

so that on RP where

$$z = 2\omega_1 - ir\varepsilon^2$$

$$4\pi\frac{\partial G}{\partial n} = -\varepsilon^2\left[\frac{p''z}{p'z-p'\bar{t}} - \frac{p''z}{p'z+p'\bar{t}}\right] + \text{complex conjugate}$$

$$= -2\left[\frac{p''ir}{p'ir-p'\bar{t}} - \frac{p''ir}{p'ir+p'\bar{t}}\right]. \tag{3.1.10}$$

Therefore $$\varphi = \int_{2\omega_1/\sqrt{3}}^{4\omega_1/\sqrt{3}}\left[2\omega_1 p(ir).\frac{3}{\pi}\left(\frac{p^2ir}{p'ir-p'\bar{t}} - \frac{p^2ir}{p'ir+p'\bar{t}}\right)\right]dr$$

i.e. $$\frac{-\pi\varphi}{6\omega_1} = \int_{2\omega_1/\sqrt{3}}^{4\omega_1/\sqrt{3}}\frac{p^3(ir)dr}{p'ir-p'\bar{t}} + \text{complex conjugate.}$$

Multiplying and dividing the integrand by $p'ir+p'\bar{t}$, we get

$$\frac{-\pi\varphi}{6\omega_1} = \frac{\omega_1}{2\sqrt{3}}p'\bar{t} - \frac{p\bar{t}}{6}\int_{2\omega_1/\sqrt{3}}^{4\omega_1/\sqrt{3}}\frac{6p^2\bar{t}(pir+p'\bar{t})}{4(p^3\bar{t}-pir)}dr + \text{complex conjugate.} \tag{3.1.11}$$

Breaking into partial fractions and employing the relations

$$p(\varepsilon z) = \varepsilon p(z),\ p(\varepsilon^2 z) = \varepsilon^2 p(z) \tag{3.1.12}$$

we get $$\frac{6p^2\bar{t}(p'ir+p'\bar{t})}{4(p^3\bar{t}-p^3ir)} = \frac{1}{2}\frac{p'\bar{t}+p'ir}{p\bar{t}-pir} + \frac{\varepsilon}{2}\frac{p'(\varepsilon\bar{t})+p'ir}{p(\varepsilon\bar{t})-pir} + \frac{\varepsilon^2}{2}\frac{p'(\varepsilon^2\bar{t})+p'ir}{p(\varepsilon^2\bar{t})-pir}. \tag{3.1.13}$$

Now $$\int\frac{1}{2}\frac{p'\bar{t}+p'ir}{p\bar{t}-pir}dr = -i\int[\zeta(\bar{t}-ir)+\zeta(ir)-\zeta(\bar{t})]d(ir)$$

$$= -i[-\log\sigma(\bar{t}-ir)+\log\sigma(ir)-ir\zeta(\bar{t})].$$

Therefore $$\int_{2\omega_1/\sqrt{3}}^{4\omega_1/\sqrt{3}}\frac{6p^2\bar{t}(p'ir+p'\bar{t})}{4(p^3\bar{t}-p^3ir)}dr = -i[-\log\sigma(\bar{t}-ir)$$

$$-\varepsilon\log\sigma(\varepsilon\bar{t}-ir)-\varepsilon^2\log\sigma(\varepsilon^2\bar{t}-ir)+\log\sigma(ir)+\varepsilon\log\sigma(ir)+\varepsilon^2\log\sigma(ir)$$

$$-ir\zeta(\bar{t})-\varepsilon ir\zeta(\varepsilon\bar{t})-\varepsilon^2 ir\zeta(\varepsilon^2\bar{t})]_{2\omega_1/\sqrt{3}}^{4\omega_1/\sqrt{3}}.$$

This expression, on employing the relations

$$\zeta(\varepsilon r) = \varepsilon^2\zeta(r),\ \zeta(\varepsilon^2 r) = \varepsilon\zeta(r) \tag{3.1.14}$$

becomes $$i\left[\log\frac{\sigma\left(\bar{t}-\frac{4i\omega_1}{\sqrt{3}}\right)}{\sigma\left(\bar{t}-\frac{2i\omega_1}{\sqrt{3}}\right)} + \varepsilon\log\frac{\sigma\left(\varepsilon\bar{t}-\frac{4i\omega_1}{\sqrt{3}}\right)}{\sigma\left(\varepsilon\bar{t}-\frac{2i\omega_1}{\sqrt{3}}\right)} + \varepsilon^2\log\frac{\sigma\left(\varepsilon^2\bar{t}-\frac{4i\omega_1}{\sqrt{3}}\right)}{\sigma\left(\varepsilon^2\bar{t}-\frac{2i\omega_1}{\sqrt{3}}\right)} + 2\sqrt{3}i\omega_1\zeta(\bar{t})\right] \tag{3.1.15}$$

Introducing the notation

$$f(z, t) = \log\sigma(z+t) + \varepsilon\log\sigma(z+\varepsilon t) + \varepsilon^2\log\sigma(z+\varepsilon^2 t). \tag{3.1.16}$$

and substituting in (3.1.11) we get

$$-\frac{\pi\varphi}{6\omega_1}=\frac{\omega_1}{2\sqrt{3}}(p'z+p'\bar{z})+\frac{\omega_1}{\sqrt{3}}(pz\zeta z+p\bar{z}\zeta\bar{z})-\frac{i\,p\,\bar{z}}{6}\left[f\left(\frac{-4i\omega_1}{\sqrt{3}},\bar{z}\right)-f\left(\frac{-2i\omega_1}{\sqrt{3}},\bar{z}\right)\right]$$

$$+\frac{i\,p\,z}{6}\left[f\left(\frac{4i\omega_1}{\sqrt{3}},z\right)-f\left(\frac{2i\omega_1}{\sqrt{3}},z\right)\right]. \tag{3.1.17}$$

Now $$f\left(\frac{2i\omega_1}{\sqrt{3}},z\right)=\log\sigma\left(\frac{2i\omega_1}{\sqrt{3}}+z\right)+\varepsilon\log\sigma\left(\frac{2i\omega_1}{\sqrt{3}}+\varepsilon z\right)+\varepsilon^2\log\sigma\left(\frac{2i\omega_1}{\sqrt{3}}+\varepsilon^2 z\right) \tag{3.1.18}$$

$\therefore$ $$\frac{df}{dz}=\zeta\left(\frac{2i\omega_1}{\sqrt{3}}+z\right)+\varepsilon^2\zeta\left(\frac{2i\omega_1}{\sqrt{3}}+\varepsilon z\right)+\varepsilon\zeta\left(\frac{2i\omega_1}{\sqrt{3}}+\varepsilon^2 z\right) \tag{3.1.19}$$

and $$\frac{d^2f}{dz^2}=-p\left(\frac{2i\omega_1}{\sqrt{5}}+z\right)-p\left(\frac{2i\omega_1}{\sqrt{3}}+\varepsilon z\right)-p\left(\frac{2i\omega_1}{\sqrt{3}}+\varepsilon^2 z\right) \tag{3.1.20}$$

$$=0$$

in virtue of the relation

$$p(z+t)=\tfrac{1}{4}\left[\frac{p'z-p't}{pz-pt}\right]^2-p(z)-p(t) \tag{3.1.21}$$

and the relation $$p(2i\omega_1/\sqrt{3})=0. \tag{3.1.22}$$

It follows that

$$f(2i\omega_1/\sqrt{3},\,z)=Az+B$$

A and B being suitable constants.

Putting $z=0$ in (3.1.18) we see that $B=0$ and by putting $z=0$ in (3.1 19), A is seen to be zero.

Therefore $$f(2i\omega_1/\sqrt{3},\,z)=0. \tag{3.1.23}$$

Similarly $$f\left(\frac{4i\omega_1}{\sqrt{3}},z\right)=f\left(-\frac{4i\omega_1}{\sqrt{3}},\bar{z}\right)=f\left(-\frac{2i\omega_1}{\sqrt{3}},\bar{z}\right)=0 \tag{3 1.24}$$

(3.1.17) therefore reduces to

$$\frac{-\pi\varphi}{6\omega_1}=\frac{\omega_1}{2\sqrt{3}}(p'z+p'\bar{z})+\frac{\omega_1}{\sqrt{3}}(pz\zeta z+p\bar{z}\zeta\bar{z}).$$

Substiuting in (3.1.3) we get

$$\frac{16\pi wD}{P}=\bar{z}pz+zp\bar{z}-\frac{\sqrt{3}\omega_1^2}{\pi}(p'z+p'\bar{z})-\frac{2\sqrt{3}\omega_1^2}{\pi}(pz\zeta z+p\bar{z}\zeta\bar{z}). \tag{3.1.25}$$

For points on QM where $z=-\varepsilon^2 r$, r denoting the distance from the origin, we write,

$$\frac{16\pi wD}{P}=-2rp(r)+\frac{4\sqrt{4}\omega_1^2}{\pi}\left(\tfrac{1}{2}p'r+pr\zeta r\right) \tag{3.1.26}$$

giving

$$\frac{wD}{P} = -2\alpha, \text{ say.}$$

Using the values of $\zeta(r, 4, 0)$ given in

TABLE 7

r	$\omega_1/4$	$\omega_1/3$	$\omega_1/2$	$2\omega_1/3$	$3\omega_2/4$
$\zeta(r)$	3·293936	2·470167	1·644647	1·225305	1·080050

we get the values of α given in

TABLE 8

r	$\omega_1/4$	$\omega_1/3$	$\omega_1/2$	$2\omega_1/3$	$3\omega_1/4$
α	·06518	·04832	·03020	·01868	·01379

Shear Forces We get from (3.1.2)

$$Q_x = \frac{-P}{4\pi}(p''z + p''\bar{z})\,;\; Q_y = \frac{-iP}{4\pi}(p''z - p''\bar{z}) \tag{3.1.27}$$

On the side QP this gives

$$Q_y = 0,\; Q_x = \frac{-3P}{\pi}p^2(y, 0, -4) \tag{3.1.28}$$

$$= \frac{-3P\alpha}{\pi}, \text{ say.}$$

The values of α for a few values of y are given in

TABLE 9

y	$\omega_1/4\sqrt{3}$	$\omega_1/2\sqrt{3}$	$3\omega_1/4\sqrt{3}$	$\omega_1/\sqrt{3}$	$3\omega_1/\sqrt{3}$
α	1059·59	66·1899	13·0026	4	0

On the median through the origin,

$$Q_x = \frac{3P}{2\pi}p^2 r,\; Q_y = \frac{3\sqrt{3}P}{2\pi}\,p^2 r \tag{3.1.29}$$

so that $$Q_r = Q_x \cos 60° + Q_y \sin 60° = \frac{3P}{\pi}p^2 r \tag{3.1.30 a}$$

$$= \frac{3P\alpha}{\pi}, \text{ say}$$

$$Q_s = 0, \tag{3·1·30 b}$$

where Q_r and Q_s have got the meaning analogous to that in the case of a square.

The values of α for a few values of r are given in

TABLE 10

r	$\omega_1/4$	$\omega_1/3$	$\omega_1/2$	$2\omega_1/3$	$3\omega_1/4$	ω_4
α	117·76	37·298	7·4641	2·5198	1·7018	1

3.2. *Couple at a point on the median through the origin.* In this case we write (1.4) in the form

$$\zeta = ip'z : g_2 = 0,\ g_3 = 4. \tag{3.2.1}$$

(1.5) now becomes

$$\nabla_1^2 w = \frac{P}{4\pi D}\left[\frac{1}{p'z-p'z_0} + \frac{1}{p'z+p'\bar{z}_0} + \frac{1}{p'\bar{z}-p'\bar{z}_0} + \frac{1}{p'\bar{z}+p'z_0}\right] \tag{3.2.2}$$

Integrating this equation and modifying the integral in such a way that we have singularities of the type $r\log r$ at the point of application of the couple and its at images in the lines QP and QR, we write

$$\frac{96wD}{P}\,p^2 r_1 = (z+r_1)\log\sigma(z+r_1) + \varepsilon(\bar{z}+\varepsilon^2 r_1)\log\sigma(z+\varepsilon r_1)$$

$$+\varepsilon^2(\bar{z}+\varepsilon r_1)\log\sigma(z+\varepsilon^2 r_1) + (z-r_1)\log\sigma(z-r_1) + \varepsilon(\bar{z}-\varepsilon^2 r_1)\log\sigma(z-\varepsilon r_1)$$

$$+\varepsilon^2(\bar{z}-\varepsilon r_1)\log\sigma(z-\varepsilon^2 r_1) + (z+r_1)\log\sigma(z+r_1) + \varepsilon(z+\varepsilon^2 r_1)\log\sigma(z+\varepsilon^2 r_1)$$

$$+\varepsilon^2(z+\varepsilon r_1)\log\sigma(z+\varepsilon r_1) + (z-r_1)\log\sigma(z-r_1) + \varepsilon(z-\varepsilon^2 r_1)\log\sigma(z-\varepsilon^2 r_1)$$

$$+\varepsilon^2(z-\varepsilon r_1)\log\sigma(z-\varepsilon r_1) + \text{complex conjugates}$$

$$+\text{a harmonic function} \tag{3.2.3}$$

where $z_0 = -\varepsilon^2 r_1$ is taken to be the point of application of the couple.

Now the expression on the right hand side of (3.2.3) excepting the harmonic function vanishes on PQ and QR, while on RP its value is found to be

$$-2\omega_1 \log \frac{(pr_1 - p\imath r)^3}{p^3 r_1 - p^3\imath r} \tag{3.2.4}$$

so that we are to find out a function which vanishes on QP and QR and assumes the value (3.2.4) on RP. (r denotes the distance from C).

The required function is found to be

$$\frac{-3\omega_1}{\eta_1}\left[\frac{\zeta z p z + \frac{1}{2}p'z}{pr_1} + \frac{\zeta z p^4 z + \frac{1}{8}p'^3 z + \frac{7}{10}p'z}{4p^4 r_1} + \ldots + \text{complex conjugates}\right]$$

$$-3\left[\frac{zp^2 z + \frac{1}{2}p'z}{2p^2 r_1} + \frac{zp^5 z + \frac{1}{8}p'^3 z + \frac{11}{14}p'z}{5p^5 r_1} + \ldots + \text{complex conjugates}\right]. \tag{3.2.5}$$

From (3.2.3) and (3.2.5) we get the required expression for w.

To take a numerical example we take $r_1 = \omega_1/2$. Writing

$$\frac{96\pi wD}{P} = \alpha$$

we have after some simplification, for points on the axis of symmetry,

$$\frac{\alpha}{p^2(\omega_1/2)} = (-4r+4r_1)\log|\sigma(r-r_1)|-(4r+4r_1)\log\sigma(r+r_1)$$
$$+(2r+4r_1)[\log(p^2r+prpr_1+p^2r_1)]$$
$$+(8r+16r_1)[\log\sigma r+\log\sigma r_1]-16r_1[\text{Real part of }\log\sigma(r+\epsilon r_1)] \quad (3.2.6)$$

r denoting the distance from the origin.

With the help of the values of real parts of $\log\sigma z$ given in

TABLE 11

z	$R\log\sigma z$	z	$R\log\sigma z$	z	$R\log\sigma z$
0	$-\infty$	$5\omega_1/6$	·0067400	$i\sqrt{3}\omega_1/4$	$\bar{1}$·3573012
$\omega_1/6$	$\bar{2}$·4024289	ω_1	·1787968	$\omega_1/3+\epsilon\omega_1/2$	$\bar{1}$·3754310
$\omega_1/4$	$\bar{2}$·8078907	$7\omega_1/6$	·3000399	$2\omega_1/3+\epsilon\omega_1/2$	$\bar{1}$·6848211
$3\omega_1/4$	$\bar{1}$·9037855	$5\omega_1/4$	·3572354	$3\omega_1/4+\epsilon\omega_1/2$	$\bar{1}$·7813824

we get the values of w given in Table 12. The quantity r in the table denotes the distance from the origin.

TABLE 12

r	$\omega_1/4$	$\omega_1/3$	$\omega_1/2$	$2\omega_1/3$	$3\omega_1/4$
α	$-$·01735	$-$·02796	·05123	·1286	·1089

It will be observed that the deflection at the point of application of the couple is not zero. However, calculations indicate that if the couple be applied at the centre of the triangle, the deflection at the centre is, perhaps, zero.

Shear Forces: We have for points on the median through the origin

$$\nabla_1^2 w = \frac{-P}{4\pi D}\left[\frac{p'r}{p^3r-p^3r_1}\right] \quad (3.2.7)$$

r denoting the distance from the origin along QM.

This gives

$$Q_s = 0\,,\ Q_r = \frac{-3P\alpha}{2\pi} \quad (3.2.8)$$

where

$$\alpha = \frac{p^2r(p^3r+p^3r_1-2)}{(p^3r-p^3r_1)^2} \quad (3.2.9)$$

The values of α corresponding to a few values of r, for $r_1 = \omega_1/2, 2\omega_1/3$ and ω_1 are given in

TABLE 13

r_1/r	$\omega_1/4$	$\omega_1/3$	$\omega_1/2$	$2\omega_1/3$	$3\omega_1/4$	ω_1
$\omega_1/2$	·00653	·21347	∞	·20999	·10622	·05157
$2\omega_1/3$	·09288	·171135	·62200	∞	2 26650	·33333
ω_1	·09222	·16446	·38490	·83995	1·39492	∞

3.3. The solution in sec. 3.2 breaks down when $r_1 = \omega_1$ for then at $ir = i\sqrt{3}\omega_1$, the expression (3.2.4) comes to possess a logarithmic singularity the series in (3.2.5) also becomes divergent at that point. But now (3.2.2) degenerates into

$$\nabla_1^2 w = \frac{P}{2\pi D}\left[\frac{1}{p'z} + \frac{1}{p'\bar{z}}\right] \tag{3.3.1}$$

giving,

$$\begin{aligned}\frac{96\pi w D}{P} &= \bar{z}\log(pz-1) + \varepsilon\bar{z}\log(pz-\varepsilon) + \varepsilon^2\bar{z}\log(pz-\varepsilon^2)\\ &\quad + z\log(p\bar{z}-1) + \varepsilon z\log(p\bar{z}-\varepsilon) + \varepsilon^2 z\log(p\bar{z}-\varepsilon^2)\\ &\quad + \text{ a harmonic function}\end{aligned} \tag{3.3.2}$$

and corresponding to (3.2.3) we may write

$$\begin{aligned}\frac{96\pi w D}{P} &= (z+\bar{z})[\log(pz-1) + \log(p\bar{z}-1)]\\ &\quad + (\varepsilon^2 z + \varepsilon\bar{z})[\log(pz-\varepsilon) + \log(p\bar{z}-\varepsilon^2)]\\ &\quad + (\varepsilon z + \varepsilon^2\bar{z})[\log(pz-\varepsilon) + \log(p\bar{z}-\varepsilon^2)]\\ &\quad + \text{ a harmonic function.}\end{aligned} \tag{3.3.3}$$

The expression on the right hand side of (3.3.3) excepting the harmonic function vanishes on the two sides through the origin and on the third side, its value is found to be

$$-2\omega_1 \log\frac{(1-p\,ir)^3}{1-p^3 ir} \tag{3.3.4}$$

exhibiting the logarithmic singularity at the point $ir = i\omega_1\sqrt{3}$.

Now consider the function

$$\frac{i}{2\pi}\left[\log\frac{p'z-2i}{p'z+2i} - \log\frac{p'\bar{z}+2i}{p'\bar{z}-2i}\right] \tag{3.3.5}$$

This is the stream function for liquid contained inside a cylinder of the equilateral triangular cross-section due to the presence of two equal and opposite sources at the points R and P, the positive one being at the point R. The sides being the stream lines, this function is constant along them. It is found that it vanishes along QP and QR and is equal to unity along RP.

Accordingly, the function

$$\frac{i}{2\pi}\log(1-p^3 z)\log\frac{p'z-2i}{p'z+2i} + \text{complex conjugate}$$

which on the boundary of the triangle is the same as

$$\frac{i}{2\pi}\log(1-p^3 z)\left[\log\frac{p'z-2i}{p'z+2i} - \log\frac{p'\bar{z}+2i}{p'\bar{z}-2i}\right]$$

vanishes along QP and QR and assumes the value $\log(1-p^3 ir)$ along RP, r denoting the distance from C.

We, therefore, modify the expression on the right of (3.3.3) by adding the term

$$\frac{2i\omega_1}{\pi}\log(1-p^3z)\log\frac{p'z-2i}{p'z+2i}+\text{ complex conjugate} \tag{3.3.6}$$

and now instead of (3·3·4) we have

$$6\omega_1\log(1-\varepsilon pir)(1-\varepsilon^2 pir)$$

which is the same as

$$6\omega_1\left[pir+\frac{p^2ir}{2}-\frac{2p^3ir}{3}+\frac{p^4ir}{4}+\ldots\ldots\right] \tag{3.3.7}$$

This series is uniformly (though conditionally) convergent at the point $pir = 1$. The term by term integration of the series obtained by multiplying (3.3.7) by Green's function for the triangle, is, therefore allowed. Consequently, the function which vanishes on QP and QR and takes on the value (3.3.7) along RP is found to be

$$
\begin{aligned}
&-6\omega_1\Big[\frac{1}{2\eta_1}\left(pz\zeta z+\tfrac{1}{2}p'z+p\bar z\zeta\bar z+\tfrac{1}{2}p'\bar z\right)\\
&+\frac{1}{4\omega_1}\left(zp^2z+\tfrac{1}{2}p'z+\bar zp^2\bar z+\tfrac{1}{2}p'\bar z\right)+\tfrac{2}{3}\Big(\frac{ip^3z}{2\pi}\log\frac{p'z-2i}{p'z+2i}\\
&-\frac{ip^3\bar z}{2\pi}\log\frac{p'\bar z+2i}{p'\bar z-2i}-\frac{p'\bar z}{2\pi}-\frac{p'z}{2\pi}\Big)+\frac{1}{8\eta_1}\left(\zeta zp^4z+\tfrac{1}{8}p'^3z+\tfrac{7}{10}p'z+\zeta\bar zp^4\bar z\right.\\
&\left.+\tfrac{1}{8}p'^3\bar z+\tfrac{7}{10}p'\bar z\right)+\frac{1}{10\omega_1}\left(zp^5z+\tfrac{1}{8}p'^3z+\tfrac{11}{14}p'z+\bar zp^5\bar z+\tfrac{1}{8}p'^3\bar z\right.\\
&\left.+\tfrac{11}{14}p'\bar z\right)+\tfrac{1}{6}\Big(\frac{ip^6z}{\pi}\log\frac{p'z-2i}{p'z+2i}-\frac{p'^3z}{4\pi}-\frac{5}{3\pi}p'z\\
&-\frac{ip^6\bar z}{\pi}\log\frac{p'\bar z+2i}{p'\bar z-2i}-\frac{p'^3\bar z}{4\pi}-\frac{5}{3\pi}p'\bar z\Big)+\ldots\ldots\Big]
\end{aligned}
\tag{3.3.8}
$$

From (3.3.3), (3.3.6) and (3.3.8) we get the desired expression for w

Shear Forces: The shear forces, at point on the axis of symmetry, are given in Table 13.

4. Right angled isosceles triangle

4.1. Couple at an angular point In this case we write (1.4) as

$$\zeta=\frac{2p^{3/2}z}{p'z},\ g_2=4,\ g_3=0. \tag{4.1.1}$$

The angular points P, Q, R of the triangle in the z-plane are taken to be $2\omega_1$, $\omega_1(1+i)$, 0, respectively; ω_1 denoting the real half period of $p(z)$. (see fig. 3)

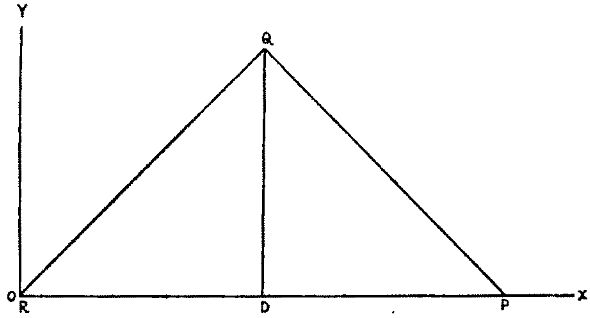


FIG. 3

(1.6) now gives
$$\frac{-2\pi iD}{P}\nabla_1^2 w = \frac{p'z}{2p^{3/2}z} - \frac{p'\bar{z}}{2p^{3/2}\bar{z}}\ . \tag{4.1.2}$$

Therefore
$$\frac{8\pi i w D}{P} = \frac{\bar{z}}{p^{\frac{1}{2}}z} - \frac{z}{p^{\frac{1}{2}}\bar{z}} + \varphi' \tag{4.1.3}$$

where φ' is harmonic.

We note to start with that
$$p^{\frac{1}{2}}(z+2\omega_1) = -p^{\frac{1}{2}}z \tag{A}$$
$$p^{\frac{1}{2}}(-z) \quad = -p^{\frac{1}{2}}z \tag{B}$$

and that branch of the function $p^{\frac{1}{2}}z$ is employed which has -1 as its residue at the origin.

It is easily seen that on RP and RQ
$$\frac{\bar{z}}{p^{\frac{1}{2}}z} - \frac{z}{p^{\frac{1}{2}}\bar{z}} = 0 \tag{4.1.4a}$$

and on PQ
$$\frac{\bar{z}}{p^{\frac{1}{2}}z} - \frac{z}{p^{\frac{1}{2}}\bar{z}} = \frac{-2\sqrt{2}i\omega_1}{p^{\frac{1}{2}}(r,-4,0)} \tag{4.1.4b}$$

r denoting the distance from the point P.

The boundary condition $w = 0$ will be satisfied, if

$$\left.\begin{aligned} &\text{on } RP \text{ and } RQ \qquad \varphi' = 0 \\ &\qquad\qquad\qquad \varphi' = \frac{2\sqrt{2}i\omega_1}{p^{\frac{1}{2}}(r,-4,0)}. \end{aligned}\right\} \tag{4.1.5}$$

We note that the requisite function φ' has got a singularity at Q, and to remove it we write,

$$\frac{8\pi i w D}{P} = \frac{\bar{z}}{p^{\frac{1}{2}}z} - \frac{z}{p^{\frac{1}{2}}\bar{z}} - \sqrt{2}\omega_1\left(\frac{e^{-\frac{1}{4}i\pi}}{p^{\frac{1}{2}}z} - \frac{e^{\frac{1}{4}i\pi}}{p^{\frac{1}{2}}\bar{z}}\right) + \varphi \tag{4.1.6}$$

where φ is harmonic.

The values of $e^{-\frac{1}{4}i\pi}/p^{\frac{1}{2}}z - e^{\frac{1}{4}i\pi}/p^{\frac{1}{2}}\bar{z}$ on RQ, PQ and RP are found to be $0, -2i/p^{\frac{1}{2}}(r, -4, 0)$ and $-\sqrt{2}i/p^{\frac{1}{2}}x$ respectively,
so that we should have on PQ and QR

$$\left.\begin{aligned} \varphi &= 0 \\ \varphi' &= \frac{-2i\omega_1}{p^{\frac{1}{2}}x} \end{aligned}\right\} \tag{4.1.7}$$

In this case Green's function $G = G(z, t)$ is given by

$$4\pi\, G = \log \frac{p^{3/2}zp't - p^{3/2}t\,p'z}{p^{3/2}zp'\bar{t} - p^{3/2}\bar{t}\,p'z} + \text{ complex conjugate.} \tag{4.1.8}$$

On RP
$$\frac{\partial}{\partial n} = -\frac{\partial}{\partial y} = -i\frac{\partial}{\partial z} + i\frac{\partial}{\partial \bar{z}},$$

so that we get after some simplification,

$$2\pi i \frac{\partial G}{\partial n} = \chi(x, t) - \chi(x, \bar{t}) \tag{4.1.9}$$

where
$$\chi(x, t) = \frac{3p^2xp'xpt + 2p^{3/2}xp'tp^{3/2}t - p'xp^3t}{2(p^3xpt - pxp^3t)}. \tag{4.1.10}$$

Therefore

$$\varphi = \int_0^{2\omega_1} \frac{-2i\omega_1}{p^{\frac{1}{2}}x}(\partial G/\partial n)\,dx = -\int_0^{2\omega_1} \frac{\omega_1}{\pi p^{\frac{1}{2}}x}[\chi(x, t) - \chi(x, \bar{t})]dx.$$

Observing that

$$p^{\frac{1}{2}}(\omega_1 + x) = p^{\frac{1}{2}}(2\omega_1 - \omega_1 + x) = -p^{\frac{1}{2}}(-\omega_1 + x) = p^{\frac{1}{2}}(\omega_1 - x) \tag{C}$$

and
$$p(\omega_1 + x) = p(\omega_1 - x) \tag{D}$$

we get
$$\frac{-\pi\varphi}{2\omega_1} = \int_0^{\omega_1} \frac{p'tp^{\frac{1}{2}}t}{p^2x - p^2t} - \text{ complex conjugate.} \tag{4.1.11}$$

We have,

$$\int_0^{\omega_1} \frac{p'tp^{\frac{1}{2}}t}{p^2x - p^2t}dx = \frac{1}{2p^{\frac{1}{2}}t}\int_0^{\omega_1} \frac{p't}{px - pt}dx + \frac{i}{2p^{\frac{1}{2}}t}\int_0^{\omega_1} \frac{p'(it)}{px - p(it)}dx$$

$$= \frac{-1}{2p^{\frac{1}{2}}t}\int_0^{\omega_1}[\zeta(x+t) - \zeta(x-t) - 2\zeta(t)]\,dx - \frac{i}{2p^{\frac{1}{2}}t}\int_0^{\omega_1}[\zeta(x+it) - \zeta(x-it) - 2\zeta(it)]dx$$

$$= \frac{-1}{2p^{\frac{1}{2}}t}\left[\log\frac{\sigma(\omega_1 + t)}{\sigma(\omega_1 - t)} - \log\frac{\sigma(t)}{\sigma(-t)} - 2\omega_1\zeta(t)\right]$$

$$-\frac{i}{2p^{\frac{1}{2}}t}\left[\log\frac{\sigma(\omega_1 + it)}{\sigma(\omega_1 - it)} - \log\frac{\sigma(it)}{\sigma(-it)} - 2\omega_1\zeta(it)\right] \tag{4.1.12}$$

The right hand, on observing that

$$\sigma(\omega_1+t) = e^{2\eta_1 t}\sigma(\omega_1-t)$$

becomes

$$\frac{-1}{2p^{\frac{1}{2}}t}\left[\log\frac{\sigma(t)}{\sigma(-t)} - i\log\frac{\sigma(it)}{\sigma(-it)}\right] + \frac{2\omega_1\zeta(t)}{p^{\frac{1}{2}}t}. \tag{4.1.13}$$

To determine the values of $\log\dfrac{\sigma(t)}{\sigma(-t)}$, $\log\dfrac{\sigma(it)}{\sigma(-it)}$, we note that, in (4.1.12) the left hand side vanishes at $t=\omega_1(1+i)$ and therefore the expression under bracket on the right hand side must also vanish. This gives, equating real and imaginary parts of this expression to zero,

$$\log\frac{\sigma(t)}{\sigma(-t)} = -\pi i;\quad \log\frac{\sigma(it)}{\sigma(-it)} = \pi i. \tag{4.1.14}$$

Accordingly, we write

$$\frac{8\pi iw\,D}{P} = \frac{\bar{z}}{p^{\frac{1}{2}}z} - \frac{z}{p^{\frac{1}{2}}\bar{z}} + \frac{4\omega_1^2}{\pi}\left[\frac{\zeta\bar{z}}{p^{\frac{1}{2}}\bar{z}} - \frac{\zeta z}{p^{\frac{1}{2}}z}\right]. \tag{4.1.15}$$

For points on the axis of symmetry, this becomes,

$$\frac{wD}{P} = -2\alpha \tag{4.1.16}$$

where

$$\alpha = \frac{(p(\omega_1-y))^{\frac{1}{2}}}{8\pi}\left[y - \frac{4\omega_1^2}{\pi}\left(\zeta y - \left(\frac{p(y)}{p(\omega_1-y)}\right)^{\frac{1}{2}}\right)\right] \tag{4.1.17}$$

the sign before each radical being now taken as positive.

Using the values of $\zeta(y, 4, 0)$ given in

TABLE 14

y	$\omega_1/4$	$\omega_1/3$	$\omega_1/2$	$2\omega_1/3$	$3\omega_1/4$
$\zeta(y)$	3·048691	2·282710	1·506642	1·098876	0·9519121

we get the values of α given in

TABLE 15

y	$\omega_1/4$	$\omega_1/3$	$\omega_1/2$	$2\omega_1/3$	$3\omega_1/4$
α	·01512	·01977	·02802	·03439	·03672

Shear Forces: From (4.1.2) we get,

$$Q_x = \frac{-iP}{\pi}\left[\frac{1}{p^{3/2}z} - \frac{4}{p^{3/2}\bar{z}}\right];\quad Q_y = \frac{P}{\pi}\left[\frac{1}{p^{3/2}z} + \frac{1}{p^{3/2}\bar{z}}\right]. \tag{4.1.18}$$

For point on the axis of symmetry, this gives,

$$Q_x = 0;\quad Q_y = \frac{2P}{\pi}p^{3/2}(\omega_1-y) = \frac{-2P\alpha}{\pi},\text{ say.} \tag{4.1.19}$$

The values of α for a few values of y are given in

TABLE 16

y	$3\omega_1/4$	$2\omega_1/3$	$\omega_1/2$	$\omega_1/3$	$\omega_1/4$
α	28·50001	12·1136	3·75114	1·77844	1·38095

4.2. Couple at a point on the axis of symmetry. It was found convenient to shift the origin in fig. 3 to the point Q and rotate the axis through 180 degrees. (4.1.1) then may be written as

$$\zeta = \frac{-ip'z}{2p^{\frac{1}{2}}z} \ ; \ g_2 = 4, \ g_3 = 0 \tag{4.2.1}$$

the branch of the function $p^{\frac{1}{2}}z$ used being one that has -1 is its residue at the origin.

(1.5) now becomes
$$\nabla_1^2 w = \frac{-P}{4\pi D}\left[\frac{p'z}{p^{\frac{1}{2}}z(p^2z-p^2z_0)} + \text{ complex conjugate}\right]. \tag{4.2.2}$$

This gives

$$\frac{-32\pi w D}{P}p^{3/2}z_0 = \bar{z}\left[\log\frac{p^{\frac{1}{2}}z-p^{\frac{1}{2}}z_0}{p^{\frac{1}{2}}z+p^{\frac{1}{2}}z_0} + i\log\frac{p^{\frac{1}{2}}z-i\,p^{\frac{1}{2}}z_0}{p^{\frac{1}{2}}z+i\,p^{\frac{1}{2}}z_0}\right] - \text{complex conjugate}$$
$$+ \text{a harmonic function} \tag{4.2.3}$$

the point z_0 being taken on the axis of symmetry.

Modifying (4.2.3) as in the previous sections, we write,

$$\frac{-32\pi w D}{P}p^{3/2}z_0 = (-z+\bar{z}+z_0-z)\log\left[p^{\frac{1}{2}}z-p^{\frac{1}{2}}z_0\right] + (z-z+z_0-z_0)\times$$
$$\log\left[p^{\frac{1}{2}}z+p^{\frac{1}{2}}z_0\right] + (iz+i\bar{z}+z_0+\bar{z}_0)\log\left[p^{\frac{1}{2}}z+p^{\frac{1}{2}}(iz_0)\right] + (-iz-i\bar{z}-z_0$$
$$+\bar{z}_0)\log\left[p^{\frac{1}{2}}z-p^{\frac{1}{2}}iz_0\right] - \text{complex conjugates} + \text{a harmonic function.} \tag{4.2.4}$$

It is found that the expression on the right hand side of (4.2.4) excepting the harmonic function vanishes on the two sides through the origin while on the side $z = -\omega_3 + x$ its value is

$$4\omega_3\log\left[\frac{p^{\frac{1}{2}}z_0-p^{\frac{1}{2}}z}{p^{\frac{1}{2}}z_0+p^{\frac{1}{2}}z}\right] + 4z_0\log\left[\frac{pz-pz_0}{piz_0-pz}\right]. \tag{4.2.5}$$

Here as before (see sec. 3 3) the function

$$4z_0\log\frac{pz-pz_0}{piz_0-pz}\cdot\frac{i}{2\pi}\log\frac{p'z+2ip^{\frac{1}{2}}z}{p'z-2ip^{\frac{1}{2}}z} - \text{complex conjugates} \tag{4.2.6}$$

vanishes on the two sides through the origin and assumes the value

$$4z_0\left[\log\frac{pz-pz_0}{piz_0-pz}\right]$$

on the third side.

As regards the term

$$4\omega_3 \log \frac{p^{\frac{1}{2}}z_0 - p^{\frac{1}{2}}z}{p^{\frac{1}{2}}z_0 + p^{\frac{1}{2}}z} \tag{4.2.7}$$

we expand it in series and considering the series term by term (which is allowed in virtue of the uniform convergence of the series at all points on PR), we get,

$$4\omega_3\Big[\frac{1}{\eta_3 p^{\frac{1}{2}}z_0}\Big(\zeta z p^{\frac{1}{2}}z + \frac{p'z}{2p^{\frac{1}{2}}z}\Big) + \frac{1}{3\omega_3 p^{3/2}z_0}\Big(zp^{3/2}z + \frac{p'z}{2p^{\frac{1}{2}}z}\Big)$$

$$+ \frac{1}{5\eta_3 p^{5/2}z_0}\Big(\zeta z p^{5/2}z + \tfrac{1}{8}\frac{p'^3z}{p^{3/2}z} + \tfrac{8}{3}\frac{p'z}{p^{\frac{1}{2}}z}\Big) + \frac{1}{7\omega_3 p^{7/2}z}\Big(zp^{7/2}z$$

$$+ \frac{p'^3z}{8p^{3/2}z} + \frac{4}{5}\frac{p'z}{p^{\frac{1}{2}}z}\Big) + \ldots\ldots\Big] - \text{complex conjugate} \tag{4.2.8}$$

as the function which vanishes on the two sides through the origin and assumes the value (4.2.7) on the third side.

From (4.2.4), (4.2.6) and (4.2.8) we get the required expression for w.

4.3. The solution in 4.2 breaks down when $z_0 = \omega_3$. But in this case (4.2.5) reduces to

$$-8\omega_3 \log(1 - ip^{\frac{1}{2}}z) + 4\omega_3 \log(p i z_0 - pz) \tag{4.3.1}$$

The series for $\log(1 - ip^{\frac{1}{2}}z)$ is uniformly (though conditionally) convergent at $ip^{\frac{1}{2}}z = -1$ (*i.e.* at the point $z = -\omega_3$). The term by term integration of the series obtained by multiplying this series by Green's function for the triangle is, therefore, allowed and consequently, we get

$$-8\omega_3\Big[\frac{i}{2\eta_3}\Big\{\zeta z p^{\frac{1}{2}}z + \frac{p'z}{2p^{\frac{1}{2}}z} + \bar{\zeta}\bar{z}p^{\frac{1}{2}}\bar{z} + \frac{p'\bar{z}}{2p^{\frac{1}{2}}\bar{z}}\Big\} + \tfrac{1}{2}\Big\{pz - \frac{p'z}{2p^{\frac{1}{2}}z} + p\bar{z} - \frac{p'\bar{z}}{2p^{\frac{1}{2}}\bar{z}}\Big\}$$

$$- \frac{i}{6\omega_3}\Big\{zp^{3/2}z + \frac{p'z}{2p^{\frac{1}{2}}z} + \bar{z}p^{3/2}\bar{z} + \frac{p'\bar{z}}{2p^{\frac{1}{2}}\bar{z}}\Big\} - \frac{1}{4\pi}\Big\{\frac{ip^2z}{2}\Big(\log\frac{p'z + 2ip^{\frac{1}{2}}z}{p'z - 2ip^{\frac{1}{2}}z} - \log\frac{p'\bar{z} - 2ip^{\frac{1}{2}}\bar{z}}{p'\bar{z} + 2ip^{\frac{1}{2}}\bar{z}}\Big)$$

$$+ \frac{p'z}{p^{\frac{1}{2}}z} + \frac{p'\bar{z}}{p^{\frac{1}{2}}\bar{z}}\Big\} + \ldots\ldots\Big] \tag{4.3.2}$$

as the function which vanishes on the two sides through the origin, and assumes the value

$$-8\omega_3 \log(1 - ip^{\frac{1}{2}}z)$$

on the third side.

Also

$$4\omega_3 \log(piz_0 - pz).\ \frac{i}{2\pi}\log\frac{p'z + 2ip^{\frac{1}{2}}z}{p'z - 2ip^{\frac{1}{2}}z} - \text{complex conjugate} \tag{4.3.3}$$

is the function which vanishes on the two sides through the origin and assume the value

$$4\omega_3 \log(piz_0 - pz)$$

on the third side.

From (4.2.4), (4.3.2) and (4.3.3) we get the required expression.

Shear Forces: Form (4.2.2) we get, for points $z = iy$ on the axis of symmetry,

$$Q_x = 0\,;\; Q_y = \frac{-2P\,p^{3/2}y}{\pi(p^2y - p^2y_1)^2}(2 - p^2y - p^2y_1) \tag{4.3.4}$$

$z_0 = iy$ being the point of application of the couple.

In the particular case when $y_1 = \omega_1$, we obtain,

$$Q_y = \frac{2P\alpha}{\pi} \tag{4.3.5}$$

where $$\alpha = p^{3/2}y/(p^2y-1). \tag{4.3.6}$$

The values of α for a few values of y are given in

TABLE 17

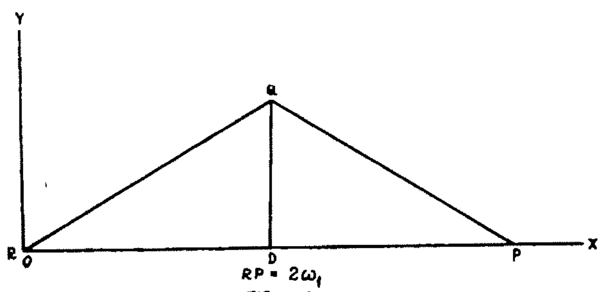
y	$\omega_1/4$	$\omega_1/3$	$\omega_1/2$	$2\omega_1/3$	$3\omega_1/4$
α	·33118	·451655	·77689	1·54018	2·56771

5. Isosceles triangle containing an angle of 120 degrees.

In this case we write

$$\zeta = \frac{2p^{3/2}z}{p'z}\,,\; g_2 = 0,\; g_3 = 4. \tag{5.1}$$

The angular points P, Q, R of the triangle in the z-plane are taken to be $2\omega_1$, $\omega_1(1+i/\sqrt{3})$, 0 respectively, ω_1 denoting the real half period $p(z)$. (see fig. 4)



RP = $2\omega_1$

FIG. 4

Now (1.6) becomes $$\nabla_1^2 w = \frac{iP}{2\pi D}\left[\frac{p'z}{2p^{3/2}z} - \frac{p'\bar{z}}{2p^{3/2}\bar{z}}\right] \tag{5.2}$$

giving $$\frac{8\pi i\,wD}{P} = \frac{z}{p^{\frac{1}{2}}z} - \frac{\bar{z}}{p^{\frac{1}{2}}\bar{z}} + \varphi' \tag{5.3}$$

where φ' is harmonic.

In this case also, the values of $p^{\frac{1}{2}}z$ on RD in the neighbourhood of R are negative and relations (A) and (B) of section 4.1 hold good.

Further, we have, on RP and RQ
$$\frac{\bar z}{p^{\frac{1}{2}}z} - \frac{z}{p^{\frac{1}{2}}\bar z} = 0 \tag{5.4 a}$$

and on PQ
$$\frac{\bar z}{p^{\frac{1}{2}}z} - \frac{z}{p^{\frac{1}{2}}\bar z} = \frac{2\omega_1 + re^{5i\pi/6}}{p^{\frac{1}{2}}(2\omega_1 + re^{5i\pi/6})} - \frac{2\omega_1 + re^{5i\pi/6}}{p^{\frac{1}{2}}(2\omega_1 + re^{-5i\pi/6})}$$

$$= \frac{-2\omega_1}{p^{\frac{1}{2}}(r, 0, -4)}(e^{5i\pi/6} - e^{-5i\pi/6})$$

$$= \frac{-2i\omega_1}{p^{\frac{1}{2}}(r, 0, -4)} \tag{5.4 b}$$

r denoting the distance from P, where use is made of the relations (A) and (B) of section (4.1) and the relation

$$p^{\frac{1}{2}}(\lambda z, \lambda^{-4}g_2, \lambda^{-6}g_3) = \lambda^{-1}p^{\frac{1}{2}}(z, g_2, g_3).$$

The boundary condition $\omega = 0$ will be satisfied, if

$$\left.\begin{aligned} &\text{on } RP \text{ and } RQ & \varphi' &= 0 \\ & & \varphi' &= \frac{2i\omega_1}{p^{\frac{1}{2}}(r, 0, -4)} \end{aligned}\right\} \tag{5.5}$$

We note that the requisite function φ' has got a singularity at Q, and to remove it, we write,

$$\frac{4\pi i w D}{P} = \frac{\bar z}{p^{\frac{1}{2}}z} - \frac{z}{p^{\frac{1}{2}}\bar z} - \frac{2\omega_1}{\sqrt{3}}\left[\frac{e^{-i\pi/6}}{p^{\frac{1}{2}}z} - \frac{e^{i\pi/6}}{p^{\frac{1}{2}}\bar z}\right] + \varphi \tag{5.6}$$

where φ is harmonic,

Now on RQ
$$\frac{e^{-\frac{1}{6}i\pi}}{p^{\frac{1}{2}}z} - \frac{e^{\frac{1}{6}i\pi}}{p^{\frac{1}{2}}\bar z} = \frac{e^{-\frac{1}{6}i\pi}}{p^{\frac{1}{2}}(r, e^{\frac{1}{6}i\pi})} - \frac{e^{\frac{1}{6}i\pi}}{p^{\frac{1}{2}}(r, e^{-\frac{1}{6}i\pi})} \tag{5.7 a}$$

and on PQ
$$\frac{e^{-\frac{1}{6}i\pi}}{p^{\frac{1}{2}}z} - \frac{e^{\frac{1}{6}i\pi}}{p^{\frac{1}{2}}\bar z} = \frac{e^{-\frac{1}{6}i\pi}}{p^{\frac{1}{2}}(2\omega_1 + re^{\frac{5}{6}i\pi})} - \frac{e^{\frac{1}{6}i\pi}}{p^{\frac{1}{2}}(2\omega_1 + re^{-\frac{5}{6}i\pi})}$$

$$= \frac{e^{-\frac{5}{3}i\pi}}{p^{\frac{1}{2}}(r, 0, -4)} + \frac{e^{-\frac{5}{3}i\pi}}{p^{\frac{1}{2}}(r, 0, -4)} = \frac{-\sqrt{3}\,i}{p^{\frac{1}{2}}(r, 0, -4)} \tag{5.7 b}$$

while on RP,
$$\frac{e^{-\frac{1}{6}i\pi}}{p^{\frac{1}{2}}z} - \frac{e^{\frac{1}{6}i\pi}}{p^{\frac{1}{2}}\bar z} = \frac{-i}{p^{\frac{1}{2}}x}, \tag{5.7 c}$$

so that we should have

on PQ and QR
$$\varphi = 0 \tag{5.8 a}$$

and on RP
$$\varphi = \frac{-2i\omega_1}{\sqrt{3}p^{\frac{1}{2}}x} \tag{5.8 b}$$

Green's function $G = G(z, t)$ is the same as in section 4, except for the changes in the values of g_2 and g_3. We get

along RP
$$\frac{\partial G}{\partial n} = X_1(x, t) - X_1(x, \bar{t}) \tag{5.9}$$

where
$$X_1(x, t) = \frac{\frac{3}{2}(p'x\,p^2x + p^{\frac{1}{2}}x\,p't\,p^{3/2}t)}{p^3x - p^3t}. \tag{5.10}$$

Therefore
$$\varphi = \int_0^{2\omega_1} \frac{-2i\omega_1}{\sqrt{3}p^{\frac{1}{2}}x}\frac{\partial G}{\partial n}\,dx$$
$$= \int_0^{2\omega_1} \frac{-\omega_1}{\pi\sqrt{3}p^{\frac{1}{2}}x}[X_1(x, t) - X_1(x, \bar{t})]dx$$

giving
$$\frac{-\pi\varphi}{\sqrt{3}\omega_1} = \int_0^{\omega_1} \frac{p't\,p^{3/2}t}{p^3x - p^3t}\,dx - \text{complex conjugate} \tag{5.11}$$

on using relations (E, F) of section 4.

Further, we have

$$\int_0^{\omega_1} \frac{p't\,p^{3/2}t}{p^3x - p^3t}\,dx = \frac{4}{3p^{\frac{1}{2}}t}\left[\log\frac{\sigma(t)}{\sigma(-t)} + \varepsilon\log\frac{\sigma(\varepsilon t)}{\sigma(-\varepsilon t)} + \varepsilon^2\log\frac{\sigma(\varepsilon^2 t)}{\sigma(-\varepsilon^2 t)} + 6\omega_1\zeta(t)\right] \tag{5.12}$$

To determine the values of $\log\frac{\sigma(t)}{\sigma(-\varepsilon t)}$, $\log\frac{\sigma(\varepsilon t)}{\sigma(-\varepsilon t)}$, $\log\frac{\sigma(\varepsilon^2 t)}{\sigma(-\varepsilon^2 t)}$ we note that the left hand side in (5.12) vanishes at the point $t = \omega_1(1 + i/\sqrt{3})$. Therefore, equating the real and imaginary parts of the expression on the right hand side to zero at that point, we get

$$\log\frac{\sigma(t)}{\sigma(-t)} = \pi i,\ \log\frac{\sigma(\varepsilon t)}{\sigma(-\varepsilon t)} = \pi i,\ \log\frac{\sigma(\varepsilon^2 t)}{\sigma(-\varepsilon^2 t)} = -\pi i. \tag{5.13}$$

From (5.6) and (5.11) we now get

$$\frac{8\pi iw\,D}{P} = \frac{\bar{z}}{p^{\frac{1}{2}}z} - \frac{z}{p^{\frac{1}{2}}\bar{z}} - \frac{2\sqrt{3}\omega_1^2}{\pi}\left(\frac{\zeta z}{p^{\frac{1}{2}}z} - \frac{\zeta\bar{z}}{p^{\frac{1}{2}}\bar{z}}\right). \tag{5.14}$$

For points on the axis of symmetry, (5·14) gives

$$\frac{wD}{P} = -2\alpha$$

where
$$\alpha = \frac{-1}{8\pi}\left[\frac{y}{p^{\frac{1}{2}}z} - \frac{6\omega_1^2}{\pi\sqrt{3}p^{\frac{1}{2}}z}\left\{\zeta(y, 0, -4) + \frac{1}{2}\frac{p'(y, 0, -4)}{1 + p(y, 0, -4)}\right\}\right]. \tag{5.15}$$

Using the values of $\zeta(y, 0, -4)$ given in

TABLE 18

y	$\omega_1/4\sqrt{3}$	$\omega_1/3\sqrt{3}$	$\omega_1/2\sqrt{3}$	$2\omega_1/3\sqrt{3}$	$3\omega_1/4\sqrt{3}$
$\zeta(y, 0, -4)$	5·705405	4·279070	2·852850	2·14016	1·90295

we get the values of α given in

TABLE 19

y	$\omega_1/4\sqrt{3}$	$\omega_1/3\sqrt{3}$	$\omega_1/2\sqrt{3}$	$2\omega_1/3\sqrt{3}$	$3\omega_1/4\sqrt{3}$
α	·0042164	·0054635	·0074834	·0085132	·008452

Shear Forces: From (5.2) we get

$$Q_x = \frac{-3iP}{2\pi}\left[\frac{1}{p^{5/2}z} - \frac{1}{p^{5/2}\bar{z}}\right], \quad Q_y = \frac{3P}{2\pi}\left[\frac{1}{p^{5/2}z} + \frac{1}{p^{5/2}\bar{z}}\right]. \tag{5.16}$$

On the axis of symmetry,

$$Q_x = 0, \qquad Q_y = \frac{-3P\alpha}{\pi}$$

where

$$\alpha = \left[\frac{p(y,0,-4)+1}{p(y,0,-4)-2}\right]^{5/2}$$

the sign before the radical being now taken as positive.

The values of α for a few values of y are given in

TABLE 20

y	$\omega_1/4\sqrt{3}$	$\omega_1/3\sqrt{3}$	$\omega_1/3\sqrt{3}$	$2\omega_1/3\sqrt{3}$	$3\omega_1/4\sqrt{3}$
α	1·26386	1·52518	2·70516	6·91254	13·93105

6. Right angled triangle containing an angle of 30 degrees.

In this case, we write. (1.4) as (Timoshenko, 1940)

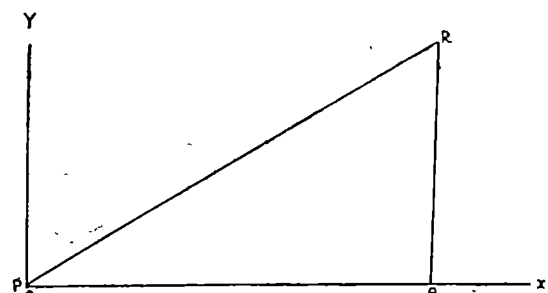
$$\zeta' = \frac{1-p^3z}{1+p^3z}\ ; g_2 = 0, \ g_3 = 4 \tag{6.1}$$

The points P, Q, R in the z-figure taken to be 0, ω_1, $\omega_1(1+i/\sqrt{3})$.

A further transformation

$$\zeta' = \frac{\zeta-1}{\zeta+1} \tag{6.2}$$

transforms the upper half plane in ζ'-plane to the same in ζ-plane. The points P, Q, R in the z-figure now correspond to 0, 1, ∞ on the real ζ-axis.



$PQ = \omega_1$

FIG. 5

(1·6) now gives
$$\nabla_1^2 w = \frac{iP}{4\pi^2 D}[p^3 z - p^3 \bar{z}] \tag{6.3}$$
so that the load is assumed to act on P.

Integration of (6.3) gives
$$\frac{-160\pi i w\, D}{P} = \bar{z}pzp'z - zpzp'\bar{z} + \varphi \tag{6.4}$$
where φ is harmonic.

$\bar{z}pzp'z - zpzp'z$ is seen to vanish on PQ and PR, while its value on QR is found to be $2\omega_1 pzp'z$.

Green's function $G = G(z, t)$, here is given by
$$4\pi G = \log \frac{p^3 z - p^3 t}{p^3 z - p^3 \bar{t}} \cdot \frac{p^3 \bar{z} - p^3 t}{p^3 \bar{z} - p^3 t} \tag{6.5}$$
so that along QR
$$4\pi \frac{\partial G}{\partial n} = \frac{6p^2 zp'z}{p^3 z - p^3 t} + \text{complex conjugate.} \tag{6.6}$$

Therefore
$$-\varphi = \int_0^{\omega_1/\sqrt{3}} 2\omega_1 pzp'z \frac{\partial G}{\partial n}\, dy$$
$$= \frac{3\omega_1}{\pi} \int_0^{\omega_1/\sqrt{3}} \frac{p^3 z p'^2 z}{p^3 z - p^3 t}\, dy - \text{complex conjugate}$$
$$= \frac{12\omega_1}{\pi} \int_0^{\omega_1/\sqrt{3}} \frac{p^6 z - p^3 z}{p^3 z - p^3 t} - \text{complex conjugate}$$
or
$$\frac{-\pi\varphi}{12\omega_1} = -i \int_{\omega_1}^{\omega_1 + i\omega_1/\sqrt{3}} \frac{(p^6 z - p^6 t) + p^6 t - (p^3 z - p^3 t) - p^3 t}{p^3 z - p^3 t}\, dz - \text{complex conjugate}$$
i.e.,
$$\frac{-\pi i \varphi}{12\omega_1} = \frac{i\omega_1}{\sqrt{3}} \cdot p^3 t - \frac{ptp't}{12}\Big[\log \frac{\sigma(\omega_1 + i\omega_1/\sqrt{3} + t)}{\sigma(\omega_1 + i\omega_1/\sqrt{3} - t)}$$
$$+ \epsilon \log \frac{\sigma(\omega_1 + i\omega_1/\sqrt{3} + \epsilon t)}{\sigma(\omega_1 + i\omega_1/\sqrt{3} - \epsilon t)} + \epsilon^2 \log \frac{\sigma(\omega_1 + i\omega_1/\sqrt{3} + \epsilon^2 t)}{\sigma(\omega_1 + i\omega_1/\sqrt{3} - \epsilon^2 t)}$$
$$- \frac{6 i \omega_1}{\sqrt{3}}\zeta(t)\Big] + \text{complex conjugate.} \tag{6.7}$$

Here also, as in section 3,
$$f(\omega_1 + i\omega/\sqrt{3}, t) = f(\omega_1 + i\omega_1/\sqrt{3}, -t) = f(\omega_1 - i\omega_1/\sqrt{3}, t) = f(\omega_1 - i\omega_1/\sqrt{3}, -t) = 0. \tag{6.8}$$
so that we may write,

$$\frac{-\pi\varphi(t, \bar{t})}{12\omega_1} = \frac{\omega_1}{\sqrt{3}}(p^3 t - p^3 t) + \frac{\omega_1}{2\sqrt{3}}[p(t)p'(t)\zeta(t) - p(t)p'(\bar{t})\zeta(\bar{t})]. \tag{6.9}$$

Substituting for φ in (6.4), we get the expression for w.

Shear forces: From $$\nabla_1^2 w = \frac{iP}{4\pi D}(p^3 z - p^3 \bar{z}) \tag{6.8 bis}$$

we get $$Q_x = \frac{-3iP}{4\pi}(p^2 z p' z - p^2 z p' z), \quad Q_y = \frac{3P}{4\pi}(p^2 z p' z + p^2 z p' z) \tag{6.10}$$

To take a numerical example, we may consider the values of shear forces on line PQ. Here, we have

$$Q_x = 0, \text{ and } Q_y = \frac{-3P\alpha}{\pi} \tag{6.11}$$

where $$\alpha = \frac{-1}{2} p^2 x p' x \tag{6.12}$$

The values of α for a few values of x are given in

TABLE 21

x	$\omega_1/4$	$\omega_1/3$	$\omega_1/2$	$2\omega_1/3$	$3\omega_1/4$
α	4207·99	561·968	32·8695	4·36449	1·87964

Similarly for points on PR, we have,

$$Q_x = \frac{3P}{4\pi} p^2(r, 0, -4) p'(r, 0, -5) \tag{6.13 a}$$

$$Q_y = \frac{-3\sqrt{3}P}{4\pi} p^2(r, 0, -4) p'(r, 0, -4)$$

r denoting the distance from P.

(6.7) gives

$$Q_{tt} = Q_x \cos 30° + Q_y \sin 30° = 0. \tag{6.14 a}$$

$$Q_{nn} = Q_x \cos 120° + Q_y \cos 30° = \frac{-3P}{2\pi} p^2(r, 0, -4) p'(r, 0, -4) = \frac{3P\alpha}{\pi}, \text{ say} \tag{6.14 b}$$

where Q_{tt} denotes the shear force per unit area across a plane tangential to the side PR while Q_{nn} across one perpendicular to PR.

The values of α for a few values of r are given in

TABLE 22

t	$\omega_1/2\sqrt{3}$	$\omega_1/3$	$\omega_1\sqrt{3}/2$
α	1537·41	12	·928204

Appendix I

Following is the method that has been adopted in deriving the numerical values of the elliptic functions and the related ones, listed in the paper.

In Table 1, the values of $p(r_1, -4, 0)$ for $r_1 = \omega_1\sqrt{3}/4, \omega_1\sqrt{2}/4, \omega_1\sqrt{2}/3$ were derived from the series

$$p(z) = \frac{1}{z^2} - \frac{z^2}{5} + \frac{z^6}{75} - \frac{2z^{10}}{3 \cdot 5^3 \cdot 13} + O(z^{14}) \tag{A.1}$$

To derive other three values we used the formula

$$p(\omega_2 + z, 4, 0) = e_2 + \frac{(e_2 - e_1)(e_2 - e_3)}{p(z, 4, 0) - e_2} = \frac{-1}{p(z, 4, 0)} \tag{A.2}$$

giving, for points on the diameter through the origin, (see sec. 2)

$$p(\omega_1\sqrt{2} - r_1, -4, 0) = [p(r_1, -4, 0)]^{-1} \tag{A.3}$$

To determine the values of $\zeta(r_1, -4, 0)$ for $r_1 = \omega_1\sqrt{2}/4, \omega_1\sqrt{2}/3$ use was made of the series

$$\zeta(z, -4, 0) = \frac{1}{z} + \frac{z^3}{15} - \frac{z^7}{525} + \frac{2z^{11}}{11 \cdot 3 \cdot 5^3 \cdot 13} + O(z^{15}). \tag{A.4}$$

The value for other values of r were derived with the help of the formula

$$\zeta(u+v) = \zeta(u) + \zeta(v) + \tfrac{1}{2}\frac{p'u - p'v}{pu - pv}. \tag{A.5}$$

Substituting $u = \omega_2$ in this, we get

$$\zeta(\omega_2 + v, 4, 0) = \eta_2 + \zeta(v, 4, 0) + \tfrac{1}{2}\frac{p'(v, 4, 0)}{p(x, 4, 0)}. \tag{A.6}$$

To determine the value of η_2, we note that

$$\eta_3\,\omega_1 - \eta_1\omega_3 = \pi i/2 \tag{A.7}$$

and since $\eta_3 = i\eta_1$, $\omega_3 = -i\omega_1$, this gives

$$\eta_3 = i\eta_1 = \frac{\pi i}{4\omega_1}. \tag{A.8}$$

Finally, $\quad -\eta_2 = \zeta(-\omega_2, 4, 0) = \zeta(\omega_1 + \omega_3, 4, 0) = \zeta(\omega_1, 4, 0) + \zeta(\omega_3, 4, 0)$

from (A.5).

Therefore
$$\eta_2 = \frac{-\pi\sqrt{2}}{4\omega_1} e^{\frac{1}{4}i\pi}. \tag{A.9}$$

For points the diameter through the origin, we write $v = r_1 e^{-\frac{1}{4}i\pi}$ in (A.6) and get

$$\zeta(\omega_1\sqrt{2}-r_1,-4,0) = \frac{\pi\sqrt{2}}{4\omega_1} - \zeta(r_1,-4,0) - \tfrac{1}{2}\frac{p'(r_1,-4,0)}{p(r_1,-4,0)}. \tag{A.10}$$

This formula determines the other three values. Thus when $r_1 = \omega_1/\sqrt{2}$ we get

$$2\zeta(\omega_1/\sqrt{2},-4,0) = \frac{\pi\sqrt{2}}{4\omega_1} + \sqrt{2} \tag{A.41}$$

and similarly for other values.

Similar considerations lead to values of $\zeta(z)$ for other values of g_2 and g_3. The first two values in every case were found from the series and the remaining three with the help of the relation (A.5) and the formula

$$\zeta(\lambda z, \lambda^{-4}g_2, \lambda^{-6}g_3) = \lambda^{-1}\zeta(z, g_2; g_3). \tag{A.12}$$

We may mention in this connection that for $g_2 = 0$, $g_3 > 0$

$$6\omega_1\eta_1 = \pi\sqrt{3}. \tag{A.13}$$

The values of $\log_e\sigma(z)$ for various values of z found from the series expansion as also (for values of arguments with modulus greater than ω_1) with the help of the formula

$$\log_e\sigma(\omega_1+z) = 2\eta_1 z + \log_e\sigma(\omega_1 - z). \tag{A.14}$$

Appendix II

When the couple is applied at an angular point of the plate, it is seen that the magnitude of the couple in the ζ-plane, and the shears and deflections at all points of the plate (exept at points very near the point of application of the couple) are all zero for all finite values of the moment of the couple in the z-plane. However, if the couple be applied at a point very near the angular point, the moment of the couple in the ζ-plane will be finite provided the magnitude of the couple in the z-plane is of the order $1/r^{n-1}$; where r represents the short distance of the point of application of the couple from the angular point and π/n is the angle there. Our solutions in such cases, therefore, represent the deflection etc. obtained by applying large couples at angular points of the plates. Also, as part of the general line of attack developed here, they illustrate the possibility of obtaining the solutions to the allied problems in the theory of bending of plates in closed form.

In this connection, we refer to the solution, given by Timoshenko (1940), for the case of a load acting at any point of a right angled isosceles triangle. It will be observed that the expression for w there, has got zeroes of order 2 and 4 at the angular points containing angles of magnitude $\pi/2$ and $\pi/4$ respectively. Since the results for the couples are obtainable from those of the load by differentiation, it serves as a verification of our results qualitatively.

It will also be observed that in these cases, the deflection at the point of application of the couple is not single-valued (except in the case of the isosceles triangle containing an angle of 120 degrees). This objection, however, does not seem to be serious. We often meet with multiple-valued expressions in literature and the only plausible reply to this objection is that our results do not hold good in the neighbourhood of the point of application of the couple.

To conclude, I wish to express my deep sense of gratitude to Prof. Dr. B. R. Seth, for his suggesting me the problem and very kind help throughout the preparation of this paper.

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References

Aggarwala, B D In a forthcoming publication in *Zeit. ang. Math. und Mech.*
Timoshenko, S. (1940) *Theory of Plates and Shells*, 295.
Love A E. H (1889) *J of Math* , **11**, 158.

THE EFFECT OF INCLINATION OF THE GEOMAGNETIC AXIS ON THE SQ. VARIATIONS

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(*Communicated by Dr. S. K. Chakrabarty—Received November 30, 1958*)

Abstract. The extent to which the Solar diurnal variation of the geomagnetic field will depend on the inclination of the Geomagnetic axis has been worked out. It is shown that this effect is very small and cannot account for the differences observed in the Sq Variations of two stations on the same latitude circle. The results obtained have been utilised in the calculation of the magnitude of the variation in the Sq field, introduced through assumed changes in the permanent magnetic field of the Earth.

The different theories put forward to explain the Sq. Variations of geomagnetic field have been worked out as a first approximation on the assumption that the phenomena depend on the geographical latitude and as such these variations are the same at all stations on the same latitude belt. Recent observations however show contrary results. Chakrabarty (1946) has shown that the Sq. variation in H at Alibag ($\varphi = 18.°6$, $\lambda = 72.°9$, $\Phi = 9.°5$) and San Juan ($\varphi = 18°4$, $\lambda = 293.°9$, $\Phi = 29.°9$) differ appreciably though the stations are situated practically on the same latitude circle. The difference is not one of amplitude only, but the general nature of the two curves are quite distinct. A second evidence for the longitudinal dependence came from the records of the observatory at Huancayo ($\varphi = -12°$, $\lambda = 283.°7$, $\Phi = -0.°6$). It is found that the H Variations at Huancayo is very much different from that at Manila ($\varphi = 14.°6$, $\lambda = 121°2$, $\Phi = 3.°3$) or at Samoa ($\varphi = -13°8$, $\lambda = 188°2$, $\Phi = -16°$). Various theories have been put forward to explain these observed results McNish (1935) has observed that the abnormality in the amplitude is evident in the Sq. variations of H while the other elements show a normal behaviour. In all previous analysis proposed to explain theoretically the geomagnetic field variations, the average values for a number of stations were used which had the obvious effect of smoothening out the anomaly referred to above. As such the previous analysis could not account for the results of observations nor even show the proper order of these anomalies. In his spherical harmonic analysis of the data of five American stations near the 245°E longitude circle, McNish (1937) has shown that the field variations have an appreciable amount of geomagnetic control. He has suggested the large divergence between the geomagnetic and geographic equators at Huancayo as a possible reason for the observed anomaly and has also remarked that a corresponding anomaly should exist on the other side of the earth, but "as the distance between the two is less there the effect is not pronounced". This anomaly has been discussed by Chapman (1948) in the light of observations made by A. Walter during his magnetic survey of Uganda and Chapman explained this by considering the station at Uganda as the northern counterpart of Huancayo as far as this phenomenon is concerned, since Uganda lies in between the geographic and magnetic equators i,e., in the "interequitorial belt".

But other stations within this belt such as Singapore ($\varphi = 1.°3$, $\lambda = 103°$) and Batavia ($\varphi = -6.°2$, $\lambda = 106.°8$) fail to show any such anomaly. Manila which is not far off from this belt also fail to exhibit such an anomalous behaviour. Chapman has also observed that the mean Uganda station is very near the geographical equator and Huancayo is very near the geomagnetic equator, and that the two observatories are situated very near the point at which the two equators cross at a very sharp angle which fact may possibly have some bearing on the anomalies exhibited at the two places.

It is therefore evident that the Sq. variation is not merely a function of geographic latitude but that it hos got a strong geomagnetic control. McNish's idea is contained in the suggestions made by Schuster (1907) that the obliquity of the magnetic axis to the geographic axis may introduce certain terms in the analysis which is dependent on the longitude of the station. Chapman (1919) has tried to treat this aspect of the problem. He obtained the coefficients of the Dynamo Current function up to a first degree of approximation but the results were inconclusive.

The Dynamo theory which explains satisfactorily the major features of different observed periodic variations as shown in previous papers is based on the nature of the field induced by the ring currents produced by the Dynamo action in the upper atmosphere. The theory also depends on the nature of the atmospheric oscillations existing in the Ionosphere, the earth's permanent magnetic field and the conductivity in the ionospheric layers. Of these the atmospheric oscillation cannot probably have a geomagnetic control, whereas the two other factors may have some geomagnetic control. Appleton (1947) has shown that the ionospheric layers in the F_2 region have a geomagnetic dependence, but the same sort of dependence is not observed in the lower layers. The height at which the ring currents, responsible for the Sq. field are produced is approximately 100 km. and is possibly within the E region, but uptodate evidences do not show any appreciable geomagntic control of the conductivity in that layer. The permanent field of the earth which is approximately that produced by a central magnetic dipole has some amount of geomagnetic control because of the existence of the inclination of the geomagnetic axis to the geographic axis. The vertical magnetic field is then approximately given by

$$H_z = C \cos\theta + C \tan\theta_0 \sin\theta \cos(\lambda - \lambda_0) \tag{1}$$

where θ, λ are the co-latitude and longitude of the station and θ_0, λ_0 are those of themagnetic north pole. The second term on the right of (1) depends on the geographic longitude of the station. In another paper (Chakrabarty & Pratap, 1953 hereafter referred as A) we have worked out the Dynamo theory and calculated the consequent geomagnetic field changes by taking for H_z only the first term in (1). It is the purpose of the present paper to examine the extent of the contribution of the second term on the right side of (1), on the strength and nature of the ring currents and their consequent effect on the Sq. variations. The results of this paper will then show the extent to which the obsrved geomagnetic control of the field variations can be associated with the earth's permanent magnetic field. The analysis followed are similar to that given in (A) Since the Dynamo equation is linear in H_z we can work out the two terms in (1) separately and then add up the result. In (A) we have

obtained the contribution of the first term. The longitude effect will therefore be obtained if we susbtitute

$$H_z = C \tan\theta_0 \sin\theta \cos\lambda_0$$

where λ_0 is the longitude of the station reckoned from the longitudinal plane containing the magnetic north pole, in the equation

$$a(\varrho e)^2\left[\frac{\partial}{\partial\varphi}vH_z + \frac{\partial}{\partial\theta}uH_z\sin\theta\right] = (\varrho e)\left[\frac{1}{\sin\theta}\frac{\partial^2 R}{\partial\varphi^2} + \frac{\partial}{\partial\theta}\sin\theta\frac{\partial R}{\partial\theta}\right]$$

$$-\left[\frac{1}{\sin\theta}\frac{\partial R}{\partial\varphi}\frac{\partial(\varrho e)}{\partial\varphi} + \sin\theta\frac{\partial R}{\partial\theta}\frac{\partial(\varrho e)}{\partial\theta}\right] \tag{2}$$

The conductivity can be expressed as a Fourier series given in (A) (Eqn: 1 to 4). Assuming the velocity potential of the atmospheric oscillations to be of the form

$$\Psi = \sum_\sigma\sum_\tau K_\sigma^\tau P_\sigma^\tau \sin[\tau t - \alpha] \tag{3}$$

the left side of the equation (2) is reduced to the form

$$K^2\sum_\sigma\sum_\tau\sum_{s=-\infty}^{\infty}\sin\theta g_s\tan\theta_0 K_\sigma^\tau\Bigg[\left\{\cos\theta\frac{dP_\sigma^\tau}{d\theta} - \sigma(\sigma+1)\sin\theta P_\sigma^\tau\right\}\tfrac{1}{2}$$

$$\times\{\sin[(s+\tau)t+\lambda_0-\alpha] + \sin[(s+\tau)t-\lambda_0-\alpha]\} - \frac{\tau P_\sigma^\tau}{2\sin\theta}\{\sin[(s+\tau)t+\lambda_0-\alpha]$$

$$-\sin[(s+\tau)t-\lambda_0-\alpha]\}\Bigg]. \tag{4}$$

The above expression consists of terms in $\sin[(s+\tau)t+\lambda_0-\alpha]$ and $\sin[(s+\tau)t-\lambda_0-\alpha]$ The current function which will satisfy the above equation should thus have two parts. Following Chapman the current function can be taken in the form

$$R = \sum_\sigma\sum_\tau CK_\sigma^\tau\tan\theta_0\sum_n\sum_m P_n^m\left[q_n^m\sin[(m+1)t-\lambda_0-\alpha] + r_n^m\sin[(m-1)t+\lambda_0-\alpha]\right] \tag{5}$$

Substituting this value of R, the right hand side of (2) reduces to

$$\sum_\sigma\sum_\tau CK_\sigma^\tau\tan\theta_0\sum_n\sum_m\sum_{s'=-\infty}^{\infty}\left[\left\{n(n+1) - \frac{ms'}{\sin^2\theta}\right\}f_sP_n^m + \frac{df_s}{d\theta}\frac{dP_n^m}{d\theta}\right]$$

$$\{q_n^m\sin[(m+s'+1)t-\lambda_0-\alpha_m] + r_n^m\sin[(m+s'-1)t+\lambda_0-\alpha_m]\}$$

or $\sum_\sigma\sum_\tau CK_\sigma^\tau\tan\theta_0\sum_n\sum_m R_n^m(s')\{q_n^m\sin[(m+s'+1)t-\lambda_0-\alpha_m] + r_n^m\sin[(m+s'-1)t+\lambda_0-\alpha_m]$ (6)

where R_n^m (s) is defined as in (A) (eqn. 15). Taking only the predominant oscillation in the upper atmosphere which is a semidiurnal one we have $\sigma = \tau = 2$. Substituting these values and after some easy reductions equation (2) is reduced to the form

$$\tfrac{2}{5}\sum_{s=-\infty}^{\infty} g_s[(9P_1^1 - 4P_3^1)\sin[(s+2)t-\lambda_0-\alpha] + 2P_3^3\sin[(s+2)t+\lambda_0-\alpha]]$$

$$= \sum_n\sum_m R_n^m(s')[q_n^m\sin[(m+s'+1)t-\lambda_0-\alpha_m] + r_n^m\sin[(m+s'-1)t+\lambda_0-\alpha_m]] \tag{7}$$

This however can be reduced to two equations, viz

$$\sum_{s=-\infty}^{\infty} \tfrac{4}{5}\, g_s P_3{}^3 \sin\left[(s+2)t+\lambda_0-\alpha\right] = \sum_n\sum_m\sum_{s'} r_n^m\, R_n^m(s') \sin\left[(m+s'-1)t+\lambda_0-\alpha_m\right] \tag{7a}$$

and

$$\sum_{s=-\infty}^{\infty} \tfrac{2}{5}\, g_s(9P_1{}^1-4P_3{}^1) \sin\left[(s+2)t-\lambda_0-\alpha\right] = \sum_n\sum_m\sum_{s'} q_n^m\, R_n^m(s') \sin\left[m+s'+1)t-\lambda_0-\alpha_m\right] \tag{7b}$$

the solution of which is also a solution of equation (7).

Equation (7) was solved exactly and uniquely for $s>-3$ assuming as in (**A**), that the relation

$$\tfrac{2}{3}a_0 a_2 - \tfrac{1}{4}a_1{}^2 = 0 \tag{8}$$

exists between the coefficients in the expression for the ionospheric conductivity. Even if this relation is not satisfied, the values of q_n^m and r_n^m thus determined will not be very much affected. The residuals for $s\leq-3$ are given in appendix A. It may be observed that the residuals are insignificant particularly when the epoch taken is the Equinox. In solving (7*b*) even assuming the relation (8) to be true the equation was solved uniquely and exactly only for $s>-2$. The coefficients are given in Tables Ia and Ib.

Chapman has assumed the conductivity to be given by the simple form $a_0+a_1\cos\omega$ in obtaining the coefficients q_n^m and r_n^m. He has also taken terms depending on cos δ and have neglected those depending on sin δ. By a simple comparison he has obtained the different terms depending on a_0 and a_1. His coefficients are also given in Table II for the equinoctial epoch for purpose of comparison with the results of our analysis. The method of successive approximations adopted by Chapman is discussed in another paper (Pratap, 1954).

TABLE Ia

Coefficients of the Current Function

$r_5{}^5 = \dfrac{1}{6300}\, a_2\gamma^2.$

$r_5{}^4 = \dfrac{1}{1575}\, a_2\beta\gamma.\ ;\quad r_4{}^4 = \dfrac{1}{280}\, a_1\gamma.$

$r_5{}^3 = \dfrac{2}{1575}\, a_2(\beta^2-\tfrac{1}{2}\gamma^2)\ ;\ r_4{}^3 = \dfrac{1}{140}\, a_1\beta.\ ;\ r_3{}^3 = \dfrac{1}{15}a_0 - \dfrac{2}{225}\, a_2\beta^2 + \dfrac{11}{450}\, a_2\gamma^2.$

$r_5{}^2 = -\dfrac{2}{525}\, a_2\beta\gamma\ ,\ r_4{}^2 = -\dfrac{1}{140}\, a_1\gamma\ ;\ r_3{}^2 = \dfrac{1}{15}a_2\beta\gamma\ ;\ r_2{}^2 = \dfrac{1}{10}a_1\dfrac{1}{\gamma} + \dfrac{3}{28}a_1\gamma$

$r_5{}^1 = \dfrac{2}{525}\, a_2\gamma^2\ ;\ r_4{}^1 = 0\ ;\ r_3{}^1 = -\dfrac{1}{15}a_2\gamma^2\ ;\ r_2{}^1 = -\dfrac{1}{5}\, a_1\dfrac{\beta}{\gamma^2}.$

$$r_1^{1} = 3a_0 - \frac{18}{5}a_0\frac{1}{\gamma^2} - \frac{12}{5}a_3\frac{1}{\gamma^2} - \frac{18}{5}a_2 + \frac{54}{35}a_2\gamma^2.$$

$$r_5^{0} = r_4^{0} = r_3^{0} = 0 \quad r_2^{0} = \frac{4}{5}a_1\frac{1}{\gamma^3} - \frac{3}{5}a_1\frac{1}{\gamma}.$$

$$r_1^{0} = -6a_0\frac{\beta}{\gamma} + \frac{72}{5}a_0\frac{\beta}{\gamma^3} + \frac{36}{5}a_2\frac{\beta}{\gamma} - \frac{48}{5}a_2\frac{\beta}{\gamma^3}.$$

$$r_5^{-1} = r_4^{-1} = r_3^{-1} = 0 \quad r_2^{-1} = \frac{12}{5}a_1\frac{\beta}{\gamma^3} - \frac{24}{5}a_1\frac{\beta}{\gamma^4}.$$

$$r_1^{-1} = \frac{18}{5}a_0 - \frac{96}{5}a_0\frac{\beta^2}{\gamma_3} + \frac{372}{35}a_2\frac{\beta^2}{\gamma^2} - \frac{72}{5}a_0\frac{\beta^2}{\gamma^4}.$$

$$-\frac{24}{5}a_0\frac{1}{\gamma^4} - \frac{48}{7}a_2\frac{\beta^2}{\gamma^4} - \frac{36}{25}\frac{a_1^2}{a_3}\frac{1}{\gamma^4} - \frac{54}{25}\frac{a_1^2}{a_2}\frac{\beta^2}{\gamma^4}.$$

$$+\frac{81}{25}\frac{a_1^2}{a_2}\frac{1}{\gamma^2} + 12\frac{a_0^2}{a_2}\frac{1}{\gamma^2} - \frac{72}{5}\frac{a_0^2}{a_2}\frac{1}{\gamma^1} - \frac{72}{35}\frac{a_1^2}{a_2} - \frac{12}{5}a_2\beta^2.$$

$$r_5^{-2} = r_4^{-2} = r_3^{-2} = 0 \quad r_2^{-2} = \frac{192}{5}a_1\frac{1}{\gamma^5} - 48a_1\frac{1}{\gamma^3} + 12a_1\frac{1}{\gamma}.$$

$$r_5^{-3} = r_4^{-3} = r_3^{-3} = 0.$$

$$r_5^{-4} = r_4^{-4} = 0.$$

$$r_5^{-5} = 0.$$

TABLE Ib

$$q_5^{3} = -\frac{1}{3150}a_2\gamma^2;\; q_4^{3} = 0 \text{ (By definition)};\; q_3^{3} = \frac{11}{900}a_2\gamma^2.$$

$$q_5^{2} = -\frac{2}{525}a_3\beta\gamma;\; q_4^{2} = -\frac{1}{140}a_1\gamma,\; q_3^{2} = \frac{2}{75}a_2\beta\gamma;\; q_2^{2} = \frac{11}{70}a_1\gamma.$$

$$q_5^{1} = -\frac{8}{525}a_2(\beta^2 - \tfrac{1}{2}\gamma^2);\; q_4^{1} = -\frac{3}{70}a_1\beta;\; q_3^{1} = -\frac{2}{15}a_0 + \frac{2}{75}a_2\beta^2 - \frac{4}{75}a_2\gamma^2.$$

$$q_2^{1} = \frac{17}{70}a_1\beta;\; q_1^{1} = 2a_0 - \frac{573}{350}a_2\beta^2 - \frac{243}{350}a_2.$$

$$q_5^{0} = \frac{8}{105}a_2\beta\gamma;\; q_4^{0} = \frac{3}{35}a_1\gamma;\; q_3^{0} = -\frac{22}{75}a_2\beta\gamma;\; q_2^{0} = -\frac{1}{5}a_1\frac{1}{\gamma} - \frac{27}{70}a_1\gamma.$$

$$q_1^{0} = -2a_0\frac{\beta}{\gamma} + \frac{12}{5}a_2\frac{\beta}{\gamma} + \frac{213}{175}a_2\beta\gamma.$$

$$q_5^{-1} = -\frac{4}{35}a_2\gamma^2,\; q_4^{-1} = 0;\; q_3^{-1} = \frac{14}{25}a_2\gamma^2;\; q_2^{-1} = \frac{6}{5}a_1\frac{\beta}{\gamma^2}.$$

$$q_1^{-1} = \frac{24}{5}a_2\frac{\beta^2}{\gamma^3} - \frac{243}{175}a_2 + \frac{9}{25}\frac{a_1^2}{a_2}\frac{1}{\gamma^2} - \frac{486}{175}a_0\frac{1}{\gamma^2}$$

$$+\frac{4}{5}\frac{a_0^2}{a_2}\frac{1}{\gamma^2} + \frac{399}{175}\frac{a_1^2}{a_2}\frac{\beta^2}{\gamma^2} - \frac{2286}{175}a_0\frac{\beta^2}{\gamma^2} + \frac{33}{50}\frac{a_1^2}{a_2}$$

$$-\frac{68}{35}a_0 + \frac{12}{25}a_2\frac{\beta^4}{\gamma^3} - \frac{108}{175}a_2\gamma^2 - \frac{51}{175}a_2\beta^2,$$

$$q_5^{-2} = q_4^{-2} = q_3^{-2} = 0. \quad q_2^{-2} = -\frac{48}{5}a_1\frac{1}{\gamma^3} + \frac{36}{5}a_1\frac{1}{\gamma}.$$

$$q_5^{-3} = q_4^{-3} = q_3^{-3} = 0.$$

TABLE II

Equinoctial values of the coefficients given in Table I with $a_0 = 1$, $a_1 = 2{\cdot}45$ and $a_2 = 9/4$ together with Chapman's values with $a_0 = 1$, $a_1 = 3$, $a_2 = 9/4$.

(a) Present paper	Chapman's	(b) Present	Chapman
= 0·000357	—	= −0·000714	—
= 0·00875	—	= 0·0275	—
= −0·00143	—	= −0·0175	—
= 0·122	0·011	= 0·385	—
= −0·0175	−0·00357	= 0·0171	—
= 0·508	0·0952	= −0·0253	−0·022
= 0·00857	—	= 0·438	0·30
= −0·15	—	= 0·210	0·025
= 0·49	—	= −1·435	−0·226
= 0·1714	—	= −0·257	—
= −2·95	—	= 1·26	—
= 5·88	—	= −6·16	—
		= −5·88	—

The additional change in H variations was calculated for Huancayo and Alibag. It is observed that the contribution is very small and amounts to only about 10 per cent. of that produced by the first term of. (1). Calculations were also made for ΔH at noon during the equinoctial season for all values of longitudes at interval of 15° for the latitude circle of Huancayo. The results show that the contribution of the second term of (1) is negligible.

Other evidences in support of the above mentioned theoretical conclusion is not lacking. Chapman while discussing Walter's observations found that Walter has failed to get any such anomaly a Mombasa which is very near to the mean Uganda station. He has concluded that the anomalous behaviour does not confine to the interequitorial region as was supposed by McNish.

Thus the only factor through which the geomagnetic control can be introduced in the Sq. variation is possibly the Ionospheric Conductivity.

I am greatly indebted to Prof. S. K. Chakrabarty for suggesting the problem and for his constant interest and encouragement in the work.

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Appendix A.

From g_{-3} group :—

In the coefficient of

$$P_2^0 = -\tfrac{2}{5}a_1a_2\beta^2\gamma - \frac{288}{35}a_1a_2\frac{1}{\gamma^3} + \frac{488}{175}a_1a_2\frac{\beta^2}{\gamma^3} + \frac{884}{175}a_1a_2\frac{1}{\gamma}$$

$$-\frac{26}{175}a_1a_2\frac{\beta^2}{\gamma} - \frac{498}{175}a_1a_2\gamma - \frac{46}{36}a_0a_1\gamma - \frac{36}{5}a_0a_1\frac{\beta^2}{\gamma} + \frac{52}{5}a_0a_1\frac{\beta^2}{\gamma^3}$$

$$+4a_0a_1\frac{1}{\gamma^3} - \frac{16}{5}a_0a_1\frac{\beta}{\gamma^3} - \frac{24}{25}a_0a_1\frac{1}{\gamma} + 2\frac{a_0^2a_1}{a_2}\frac{1}{\gamma} - \frac{12}{5}\frac{a_0^2a_1}{a_2}\frac{1}{\gamma^3}.$$

$$P_1^0 = \frac{216}{5}a_0^2\frac{\beta}{\gamma^3} - 24a_0^2\frac{\beta}{\gamma} + \frac{72}{35}a_0a_2\frac{\beta}{\gamma^3} - \frac{8}{5}a_0a_2\beta\gamma + \frac{708}{35}a_0a_2\frac{\beta}{\gamma}$$

$$+\frac{96}{5}a_0a_2\frac{\beta^3}{\gamma} + \frac{216}{5}a_0a_2\frac{\beta^3}{\gamma^3} + \frac{72}{35}a_1^2\beta\gamma - \frac{72}{35}a_1^2\beta\gamma - \frac{88}{35}a_2^2\frac{\beta}{\gamma}$$

$$-\frac{372}{35}a_2^2\frac{\beta^3}{\gamma} + \frac{48}{7}a_2^2\frac{\beta^3}{\gamma^3}.$$

$$P_0^0 = -\frac{6}{25}a_1a_2\gamma + \frac{2}{5}a_1a_2\beta^2\gamma + \frac{192}{25}a_1a_2\frac{1}{\gamma^3} - \frac{176}{25}a_1a_2\frac{1}{\gamma}$$

$$-\frac{136}{25}a_1a_2\frac{\beta^2}{\gamma} + \frac{704}{175}a_1a_2\frac{\beta^2}{\gamma^3} + \frac{24}{5}a_0a_1\frac{\beta^2}{\gamma} + \frac{816}{25}a_0a_1\frac{\beta^2}{\gamma^3}$$

$$-\frac{264}{25}a_0a_1\frac{1}{\gamma} + \frac{184}{35}a_0a_1\gamma + \frac{144}{25}a_0a_1\frac{1}{\gamma^3} - 8\frac{a_0^2a_1}{a_2}\frac{1}{\gamma}$$

$$+\frac{48}{5}\frac{a_0^2a_1}{a^2}\frac{1}{\gamma^3}.$$

From g_{-4} group :—

In the coefficient of

$$P_4^{-1} = \frac{192}{35}a_1a_2\frac{\beta^3}{\gamma_4} + \frac{36}{35}a_1a_2\beta.$$

$$P_2^{-1} = -\frac{792}{35}a_1a_2\beta + 6a_0a_1\beta - \frac{144}{5}a_0a_1\frac{\beta}{\gamma 4} + \frac{2664}{35}a_1a_2\frac{\beta}{\gamma^2}$$

$$-\frac{864}{35}a_1a_2\frac{\beta}{\gamma_4} + r_1^{-1}a_1\beta.$$

$$P_1^{-1} = r_1^{-1}(2a_0 + a_2\gamma^2) + \frac{96}{35}a_2^2\beta^2\gamma^2 - \frac{12}{35}a_2^2\gamma^2 - \frac{72}{35}a_2^2\beta^2 - \frac{16}{7}a_2^2\frac{\beta^2}{\gamma^2}$$

$$- \frac{297}{25}a_1^2 + \frac{216}{5}a_1^2\frac{1}{\gamma^2} - \frac{864}{25}a_1^2\frac{1}{\gamma^4} + \frac{108}{25}a_1^2\frac{\beta^2}{\gamma^2} - \frac{216}{25}a_1^2\frac{\beta^2}{\gamma^4}$$

$$+ \frac{36}{25}a_1^2\frac{1}{\gamma^2} - 2a_0a_2\beta^2 + \frac{6}{5}a_0a_2 - \frac{48}{5}a_0a_2\frac{\beta^2}{\gamma^2}.$$

From g_{-5} group.

In the coefficient of

$$P_4^{-2} = \frac{1584}{35}a_1a_2\frac{1}{\gamma} - \frac{468}{35}a_1a_2\gamma + \frac{2304}{35}a_1a_2\frac{\beta^2}{\gamma^5} - \frac{1728}{35}a_1a_2\frac{\beta^2}{\gamma^3} + \frac{144}{35}a_1a_2\frac{\beta^2}{\gamma} - \frac{1152}{35}a_1a_2\frac{1}{\gamma}$$

$$P_3^{-2} = \frac{2304}{5}a_1^2\frac{\beta}{\gamma^5} - \frac{384}{5}a_1^2\frac{\beta}{\gamma^3} + \frac{48}{5}a_1^2\frac{\beta}{\gamma}$$

$$P_2^{-3} = -r_1^{-1}2a_1\gamma + \frac{1308}{35}a_1a_2\gamma + \frac{384}{35}a_1a_2\frac{\beta^2}{\gamma^5} + \frac{384}{35}a_1a_2\frac{\beta^2}{\gamma^3} - \frac{312}{35}a_1a_2\frac{\beta^2}{\gamma}$$

$$+ \frac{768}{7}a_1a_2\frac{1}{\gamma^3} - \frac{4944}{35}a_1a_2\frac{1}{\gamma}.$$

From g_{-6} group:—

In the coefficient of

$$P_4^{-3} = 288a_1a_2\frac{\beta}{\gamma^3} - \frac{1152}{5}a_1a_2\frac{\beta}{\gamma^4} - \frac{372}{5}a_1a_2\beta$$

$$P_3^{-3} = -\frac{2304}{5}a_1^2\frac{1}{\gamma^3} + 576a_1^2\frac{1}{\gamma^2} - 144a_1^2.$$

From g_{-7} group.

In the coefficient of

$$P_4^{-4} = \frac{4608}{5}a_1a_2\frac{1}{\gamma^3} - 1152a_1a_2\frac{1}{\gamma} + 288a_1a_2\gamma$$

Appendix B.

From g_2 group:—

In the coefficient of $P_3^3 = \frac{1}{50}a_2^2\beta^2\gamma^2 + \frac{1}{100}a_2^2\gamma^4$.

From g_1 group :—

In the coefficient of $P_3^2 = \frac{1}{50} a_2^2 \beta \gamma^3 + \frac{2}{25} a_2^2 \beta^3 \gamma$.

From g_0 group :—

In the coefficient of $P_3^1 = \frac{4}{25} a_2^2 \beta^4 - \frac{4}{25} a_2^2 \beta^2 \gamma^2 + \frac{1}{50} a_2^2 \gamma^4$.

From g_{-1} group :—

In the coefficient of $P_3^0 = \frac{6}{25} a_2^2 \beta \gamma^3 - \frac{24}{25} a_2^2 \beta^3 \gamma$.

$$P_2^0 = \frac{1}{6} a_1 \gamma q_1^{-1} - \frac{3}{5} a_1 a_2 \gamma^3 - \frac{123}{175} a_1 a_2 \beta^2 \gamma + \frac{228}{175} a_1 a_2 \frac{1}{\gamma} + \frac{5}{3} a_0 a_1 \gamma - \frac{359}{350} a_1 a_2 \gamma - \frac{38}{15} a_0 a_1 \frac{1}{\gamma}.$$

$$P_1^0 = -q_1^{-1} a_2 \beta \gamma + 2 a_2 \beta \gamma q_1^1 - \frac{9}{5} a_1^2 \frac{\beta}{\gamma} - 4 a_0^2 \frac{\beta}{\gamma} - \frac{3}{2} a_1^2 \beta \gamma + \frac{4}{5} a_0 a_2 \frac{\beta}{\gamma}$$

$$+ \frac{2978}{525} a_0 a_2 \beta \gamma - \frac{16}{5} a_2^2 \beta \gamma + \frac{162}{175} a_2^2 \beta^3 \gamma + \frac{26}{35} a_2^2 \beta \gamma^3 + \frac{24}{5} a_2^2 \frac{\beta}{\gamma}$$

$$P_0^0 = -\frac{12}{5} a_0 a_1 \gamma - \frac{2}{3} a_1 \gamma q_1^{-1} + \frac{4}{3} a_1 \beta q_1^0 + \frac{4}{3} a_1 \gamma q_1^1 - \frac{87}{175} a_1 a_2 \gamma^3$$

$$- \frac{132}{17} a_1 a_2 \beta^2 \gamma - \frac{16}{5} a_1 a_2 \frac{1}{\gamma} + \frac{44}{25} a_1 a_2 \gamma - \frac{32}{25} a_1 a_2 \frac{\beta^2}{\gamma}.$$

From g_{-2} group :—

In the coefficient of $P_4^{-1} = \frac{24}{35} a_1 a_2 \beta - \frac{48}{35} a_1 a_2 \frac{\beta}{\gamma^2}$.

$$P_3^{-1} = -\frac{6}{25} a_2^2 \gamma^4 + \frac{84}{25} a_2^2 \beta^2 \gamma^2.$$

$$P_2^{-1} = \frac{36}{175} a_1 a_2 \beta \gamma^2 - \frac{36}{7} a_1 a_2 \beta + \frac{36}{5} a_0 a_1 \frac{\beta}{\gamma^2} + \frac{36}{5} a_0 a_1 \beta + \frac{24}{7} a_1 a_2 \beta^3 + a_1 \beta q_1^{-1} - a_1 \gamma q_1 .$$

$$P_1^{-1} = (2a_0 + \tfrac{3}{5} a_2 \gamma^2) q_1 - \frac{129}{50} a_1^2 \gamma^2 + \frac{54}{5} a_1^2 \frac{1}{\gamma^2} - \frac{27}{5} a_1^2 - \frac{4}{3} a_0 a_2 \beta^2$$

$$+ \frac{8}{5} a_2^2 \beta^2 - \frac{36}{35} a_0 a_2 \gamma^2 + \frac{54}{25} a_1^2 \beta^2 + \frac{967}{175} a_2 \gamma^2 \beta^2 + \frac{256}{175} a_2^2 \gamma^4 + \frac{243}{175} a_2^2 \gamma^2.$$

From g_{-3} group :—

In the coefficient of

$$P_3^{-2} = \frac{18}{5} a_1^2 \frac{\beta}{\gamma} - \frac{36}{5} a_2^2 \beta \gamma^3 + \frac{32}{5} a_1^2 \beta \gamma - \frac{192}{5} a_1^2 \frac{\beta}{\gamma^3}.$$

$$P_2^{-2} = 24a_0a_1\gamma + \frac{96}{5}a_0a_1\frac{1}{\gamma} - \frac{288}{5}a_0a_1\frac{1}{\gamma^3} - \frac{176}{35}a_1a_2\beta^2\gamma - \frac{184}{35}a_1a_2\frac{\beta^2}{\gamma} - \frac{96}{35}a_1a_2\frac{\beta^2}{\gamma^3}$$

$$- \frac{192}{35}a_1a_2\frac{1}{\gamma} + \frac{284}{35}a_1a_2 + \frac{3638}{1225}a_1a_2\gamma^3 - 2a_1\gamma q_1^{-1}.$$

From g_{-4} group :— $P_4^{-3} = \frac{576}{5}a_1a_2\frac{\beta}{\gamma^2} - \frac{168}{5}a_1a_2\beta\gamma^2 - 24a_1a_2\beta.$

From g_{-5} group $P_4^{-4} = 96a_1a_2\gamma^3 - \frac{1152}{5}a_1\frac{1}{\gamma} + \frac{384}{5}a_1a_2\gamma.$

References

Appleton, (1919) *Phil. Trans. Roy. Soc A*. **218**, 1.
Chapman, S. (1919) *Phil. Trans. Roy. Soc.*, A. **208**. 1.
Chapman, S. (1948) *Terr. Mag. & Atm. Elec.* **53**, 247.
Chakrabarty, S. K. (1946) *Curr. Sci.* **15**, 246.
Chakrabarty and Pratap. (1954) *Journal of Geophysical Research* **56**, 1.
McNish (1935) *Terr. Mag. & Atm. Elec* **40**, 151.
——— (1937) *I.A.T.M.E. Bull* **10**, *Transactions of*, 271.
Pratap. To be published.
Schuster, A. (1907) *Phil. Trans. Roy. Soc. A* **208**, 160.

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A NOTE ON JEFFERY'S EXACT SOLUTION OF STEADY TWO-DIMENSIONAL MOTION OF A VISCOUS LIQUID

By

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(*Communicated by Dr. S. Ghosh—Received March 29, 1954.*)

1. Jeffery (1915) has obtained an exact solution of the two-dimensional equations of motion of a viscous liquid by assuming that lines of equal vorticities are harmonic curves. He has proved that this can only happen when the lines of equal vorticities are concentric circles. By following a more profound method Hamel (1917) has obtained another class of exact solutions by assuming that the stream lines are harmonic curves. As Jeffery has, in his demonstration, stated without proof an important result (communicated to him by Watson) and as very few exact solutions of two-demensional motion of a viscous liquid are known, it is not without interest to obtain Jeffery's result by following the more fruitful method of Hamel. The object of the present note is to supply such a proof which is free from any assumption.

2. Using curvilinear orthogonal coordinates

$$\zeta = \xi + i\eta = f(z) = f(x+iy) \tag{1}$$

Hamel has shown that the vorticity Ω and stream function ψ satisfy the equations

$$\nu\Delta'\Omega = \frac{\partial(\psi, \Omega)}{\partial(\xi, \eta)} \tag{2}$$

$$\frac{\partial(\psi, \Delta'\psi)}{\partial(\xi, \eta)} - \Delta'\psi\left(b\frac{\partial\psi}{\partial\xi} + a\frac{\partial\psi}{\partial\eta}\right) = \nu\left[\Delta'\Delta'\psi + 2\left(a\frac{\partial\Delta'\psi}{\partial\xi} - b\frac{\partial\Delta'\psi}{\partial\eta}\right) + (a^2+b^2)\,\Delta'\psi\right] \tag{3}$$

where $$a = \partial \log Q/\partial\xi, \quad b = -\partial \log Q/\partial\eta, \quad Q = |d\zeta/dz|^2$$

and $$\Delta' = \frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}.$$

The mean pressure at any point is given by

$$\frac{p}{\varrho} + \tfrac{1}{2}v^2 = \int\left[\nu\left(-\frac{\partial\Delta\psi}{\partial\eta}\,d\xi + \frac{\partial\Delta\psi}{\partial\xi}\partial\eta\right) + \Delta\psi d\psi\right] \tag{4}$$

3. Taking $$\Omega = F(\xi)$$

and substituting in (2) we get

$$\frac{\partial\psi}{\partial\eta} = -\nu\,\frac{F''(\xi)}{F'(\xi)}, \tag{5}$$

assumming $F'(\xi)\neq 0$, that is, the vorticity is not constant. Operating on both sides of (5) by Δ' and substituting

$$\Delta'\psi = 2F(\xi)/Q,$$

we get

$$\frac{\partial}{\partial\eta}\left(\frac{1}{Q}\right) = -\frac{\nu}{2F}\frac{d^2}{d\xi^2}\left\{\frac{F''(\xi)}{F'(\xi)}\right\} = f(\xi), \text{ say,} \tag{6}$$

which gives on integration

$$1/Q = \eta f(\xi)+g(\xi). \tag{7}$$

Since $\log Q$ is a plane harmonic function we see that f and g satisfy the three non-linear equations

$$f'^2-ff'' = 0, \tag{8}$$

$$2f'g'-fg''-gf'' = 0. \tag{9}$$

$$f^2+g'^2-gg'' = 0. \tag{10}$$

An integral of (8) is obtained as

$$f = Ae^{p\xi}$$

Substituting in (9) we get the equation for g as

$$g''-2pg'+p^2g = 0$$

which gives
$$g = (B\xi+C)e^{p\xi}.$$

Substituting for f and g in (10) we get

$$A^2+B^2 = 0.$$

Since f and g are real we have $A = 0$, $B = 0$; therefore the solutions of the equations (8), (9), (10) are given by

$$f = 0, \quad g = ce^{p\xi}. \tag{11}$$

Hence
$$1/Q = ce^{p\xi}, \; a = -p, \; b = 0.$$

Therefore
$$\xi = -2/a \log z,$$

except in the degenerate case $a = 0$

This shows that lines of equal vorticities ξ = constant are concentric circles. Since Q is a function of ξ alone, equation (6) gives

$$F''/F' = D\xi+E$$

and the equation (5) gives

$$\psi = -\nu(D\xi+E)\,\eta+h(\xi). \tag{12}$$

As ψ satisfies Hamel's equation (3) in which $b = 0$, a = constant, we see that h satisfies the equation

$$(D\xi+E)h'''+a(D\xi+E)h'' = h^{iv}+2ah'''+a^2h''. \tag{13}$$

Finally

$$\frac{p}{\varrho}+\tfrac{1}{2}v^2 = -\frac{4\nu}{a^2}\eta e^{a\xi}[h'''+(a-E-D\xi)h'']+\frac{4}{a^2}\int e^{a\xi}h'h''d\xi. \tag{14}$$

For the pressure to be single-valued, we must have

$$h'''+(a-E-D\xi)h'' = 0. \tag{15}$$

A solution of (13) is given by

$$h = c_1+c_2\xi+c_3 \exp[(E-a)\xi]+c_4 \exp(-a\xi)$$

when $D=0$, $E\neq a$,

$$h = c_1+c_2\xi+c_3\xi^2+c_4 \exp(-a\xi)$$

when $D=0$, $E=a$ and

$$h = b_1+b_2\xi+b_3 \exp(-a\xi)+b_4 \int d\xi \int \exp(-a\xi)d\xi \int \exp(\tfrac{1}{2}D\xi^2)d\xi$$

when $E=0$, $D\neq 0$.

These are equivalent to Jeffery's solution.

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References.

Hamel, G. (1917), *Deut. Math. Ver*, 25, 35.
Jeffery, G. B. (1915), *Phil. Mag.*, (6), 29, 455.

TWO-DIMENSIONAL BOUNDARY LAYER FLOW ALONG A WALL IN A CONVERGING CHANNEL WITH CURVED BOUNDARIES

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(*Communicated by Dr. S. Ghosh—Received March 29, 1954*)

Introduction. The only problem of two-dimensional boundary layer flow along a wall of a converging channel that has been treated so far is the problem in which the flow takes place between two converging straight walls. Pohlhausen (1921) has given an exact solution of the boundary layer equations in this case by assuming that the irrotational flow outside the boundary layer is the same as if there were a source or a sink at the point towards which the walls converge. In the present paper the flow in the boundary layer in a converging channel with two circular boundaries, or with a circular and a straight boundary is considered. It is assumed that when the boundaries are produced they touch each other. In this case the irrotational flow outside the boundary layer can be considered as due to a doublet of suitable strength and orientation at the point of contact. The boundary layer equations along a circular boundary are deduced from the Stokes-Navier equations and the equation of continuity. As an exact solution of these equations can not be obtained the approximate method of Pohlhausen has been made use of to find the thickness of the boundary layer at any point of the wall. The boundary layer equations are at first integrated through the boundary layer and put in a form suitable for the application of the Pohlhausen method.

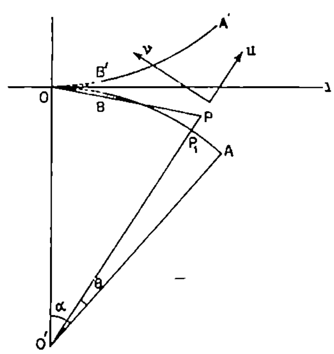

FIGURE 1

Boundary Layer Equations We consider the steady two-dimensional motion of an incompressible viscous fluid in the converging channel formed by the two circular arcs AB and $A'B'$ which when produced touch at O. Let OX be the common tangent at O. To consider the boundary layer on the wall AB we take the centre O' of the arc AB as pole and the radius $O'A$ through the point A where the fluid enters the channel as the

initial line. Let the angle $OO'A = \alpha$. We assume that the fluid enters the channel at A tangentially to the arc AB. Let (r, θ) be the polar coordinates of a point P of the fluid in the channel. To determine the steady two-dimensional irrotational flow in the channel, we observe that since the two circular arcs touching each other at O are stream lines the motion is the same as if there were a doublet at O with its axis along OX. The velocity at P due to this doublet is $2am/OP^2$. where a is the radius of the circular arc AB and m the quantity of fluid that flows between AB and XO in unit time. Hence the velocity in the irrotational flow at a point on AB is given by

$$q = m/[2a \sin^2\tfrac{1}{2}(\alpha-\theta)]$$

and the pressure is given by

$$p/\varrho = \text{constant} - m^2/[8a^2 \sin^4\tfrac{1}{2}(\alpha-\theta)].$$

Therefore

$$\frac{1}{\varrho}\frac{\partial p}{\partial \theta} = -\frac{m^2 \cot\frac{1}{2}(\alpha-\theta)}{4a^2 \sin^4\frac{1}{2}(\alpha-\theta)}. \tag{1}$$

If u, v be the components of velocity in the directions of r and θ, the equations of motion in the boundary layer are (Goldstein, 1950, p. 104)

$$u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} - \frac{v^2}{r} = -\frac{1}{\varrho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 u - \frac{u}{r^2} - \frac{2}{r^2}\frac{\partial v}{\partial \theta}\right) \tag{2}$$

$$u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + \frac{uv}{r} = -\frac{1}{\varrho}\frac{1}{r}\frac{\partial p}{\partial \theta} + \nu\left(\nabla^2 v + \frac{2}{r^2}\frac{\partial u}{\partial \theta} - \frac{v}{r^2}\right) \tag{3}$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

The equation of continuity is

$$\frac{\partial u}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r} = 0. \tag{4}$$

The boundary conditions are

$$u = 0, \quad v = 0 \text{ when } r = a$$

$$v = m/[2a \sin^2\tfrac{1}{2}(\alpha-\theta)],\ \partial v/\partial r = 0 \text{ when, } r = a+\delta. \tag{5}$$

Assuming $v, \dfrac{1}{r}\dfrac{\partial v}{\partial \theta} = O(1)$ the equation of continuity shows that $u = O(\delta)$. Neglecting u/r in comparison with other terms the approximate equation of continuity is

$$\frac{\partial u}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta} = 0. \tag{6}$$

The left hand side of equation (3) is atmost of order 1 while the greatest term in the second member on the right hand side of (3) is $\nu\partial^2 v/\partial r^2$. Therefore if the motion is to be influenced by viscosity this term must be of $O(1)$. But as $\partial^2 v/\partial r^2$ is of $O(1/\delta^2)$, ν must be of order δ^2, or $\delta = O(\nu^{\frac{1}{2}})$.

The second equation of motion becomes therefore

$$u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} = -\frac{1}{\varrho r}\frac{\partial p}{\partial \theta} + \nu\frac{\partial^2 v}{\partial r^2}. \tag{7}$$

Retaining only terms of $O(1)$ in the equation (2) we get

$$\frac{v^2}{r} = \frac{1}{\varrho}.\frac{\partial p}{\partial r}. \tag{8}$$

Therefore the variation of pressure throughout the boundary layer along a normal to the wall is of order δ and may therefore be neglected. Equations (6) and (7) are the boundary layer equations of this problem. In (7), $\partial p/\partial\theta$ is to be replaced by the value given by (1).

It seems unlikely that an exact solution of these equations could be obtained. We therefore follow the approximate method of Pohlhausen, which gives results in close agreement with experiment, when the flow is in the direction of the pressure gradient, as is the ca e here.

Boundary Layer Momentum Integral. To obtain the momentum integral in the boundary layer we integrate the equati n (7) with respect to r between the limits $r = a$ and $r = a+\delta$, where δ is the thickness of the boundary layer on it. We get on using the boundary conditions (5)

$$\int_a^{a+\delta} u\frac{\partial v}{\partial r}dr + \int_a^{a+\delta} \frac{v}{r}\frac{\partial v}{\partial \theta}dr = \frac{m^2 \cot\frac{1}{2}(\alpha-\theta)}{4a^2\sin^4\frac{1}{2}(\alpha-\theta)}\int_a^{a+\delta}\frac{dr}{r} - \nu\left(\frac{\partial v}{\partial r}\right)_{r=a} \tag{9}$$

Integrating the first integral on the left hand side by parts and using (5) and (6) we get

$$\int_a^{a+\delta} u\frac{\partial v}{\partial r}dr = uv\Big|_a^{a+\delta} - \int_a^{a+\delta} v\frac{\partial u}{\partial r}dr = \frac{m}{2a\sin^2\frac{1}{2}(\alpha-\theta)}(u)_{r=a+\delta} + \int_a^{a+\delta}\frac{v}{r}\frac{\partial v}{\partial\theta}dr \tag{10}$$

But
$$(u)_{r=a+\delta} = \int_a^{a+\delta}\frac{\partial u}{\partial r}dr = -\int_a^{a+\delta}\frac{1}{r}\frac{\partial v}{\partial\theta}dr$$

'rom (6); so that (10) becomes

$$-\frac{m}{2a\sin^2\frac{1}{2}(\alpha-\theta)}\int_a^{a+\delta}\frac{1}{r}\frac{\partial v}{\partial\theta}dr + \int_a^{a+\delta}\frac{v}{r}\frac{\partial v}{\partial\theta}dr. \tag{11}$$

Substituting in (9) and neglecting the variation of r within the boundary layer we get

$$-\frac{m}{2a^2\sin^2\frac{1}{2}(\alpha-\theta)}\int_a^{a+\delta}\frac{\partial v}{\partial\theta}dr + \frac{2}{a}\int_a^{a+\delta} v\frac{\partial v}{\partial\theta}dr = \frac{m^2\cot\frac{1}{2}(\alpha-\theta)}{4a^3\sin^4\frac{1}{2}(\alpha-\theta)}\int_a^{a+\delta}dr - \nu\left(\frac{\partial v}{\partial r}\right)_{r=a} \tag{12}$$

This is the required momentum integral of the boundary layer.

Application of the Pohlhausen Method. For the application of the Pohlhausen method we have to assume a plausible distribution of the velocity v satisfying the boundary conditions (5). If we take

$$v = \frac{m\varphi(\eta)}{2a\sin^2\frac{1}{2}(\alpha-\theta)} \tag{13}$$

where
$$\varphi(\eta) = 2\eta-\eta^2,\ \eta = (r-a)/\delta \tag{14}$$

we see that
$$v = 0 \text{ when } \eta = 0$$

$$v = \frac{m}{2a \sin^2 \frac{1}{2}(\alpha-\theta)}, \quad \frac{\partial v}{\partial \eta} = 0 \quad \text{when} \quad \eta = 1$$

These are equivalent to the boundary conditions for v given in (5). This expression for v gives

$$\left.\begin{aligned} \frac{\partial v}{\partial \theta} &= \frac{m}{2a \sin^3 \frac{1}{2}(\alpha-\theta)}\left[\varphi \cot \tfrac{1}{2}(\alpha-\theta) - \frac{\eta}{\delta}\frac{\partial \varphi}{\partial \eta}\frac{d\delta}{d\theta}\right] \\ v\frac{\partial v}{\partial \theta} &= \frac{m^2}{4a^2 \sin^4 \frac{1}{2}(\alpha-\theta)}\left[\varphi^2 \cot \tfrac{1}{2}(\alpha-\theta) - \frac{n}{\delta}\,\varphi\,\frac{\partial \varphi}{\partial \eta}\frac{d\delta}{d\theta}\right] \end{aligned}\right\} \tag{15}$$

Introducing the variable η given by (14) equation (12) becomes

$$-\frac{m\delta}{2a^2 \sin^2 \frac{1}{2}(\alpha-\theta)}\int_0^1 \frac{\partial v}{\partial \theta}d\eta + \frac{2\delta}{a}\int_0^1 v\frac{\partial v}{\partial \theta}d\eta = \frac{m^2\delta \cot \frac{1}{2}(\alpha-\theta)}{4a^3 \sin^4 \frac{1}{2}(\alpha-\theta)}\int_0^1 d\eta - \frac{\nu}{\delta}\left(\frac{\partial v}{\partial \eta}\right)_{\eta=0} \tag{16}$$

Substituting the values of $\frac{\partial v}{\partial \theta}, v\frac{\partial v}{\partial \theta}$ from (15) in this equation and observing that

$$\int_0^1 \eta\frac{\partial \varphi}{\partial \eta}d\eta = 1 - \int_0^1 \varphi d\eta \quad \text{and} \quad \int_0^1 \eta\varphi\frac{\partial \varphi}{\partial \eta}d\eta = \tfrac{1}{2}\left(1 - \int_0^1 \varphi^2 d\eta\right)$$

we get

$$\frac{m\delta}{4a^3 \sin^4 \frac{1}{2}(\alpha-\theta)}\left[-\cot \tfrac{1}{2}(\alpha-\theta)\int_0^1 \varphi d\eta + \frac{1}{\delta}\frac{d\delta}{d\theta}\left(1 - \int_0^1 \varphi d\eta\right)\right.$$
$$\left. + 2\cot \tfrac{1}{2}(\alpha-\theta)\int_0^1 \varphi^2 d\eta - \frac{1}{\delta}\frac{d\delta}{d\theta}\left(1 - \int_0^1 \varphi^2 d\eta\right) - \cot \tfrac{1}{2}(\alpha-\theta)\right]$$
$$= -\frac{\nu m}{a\delta \sin^2 \frac{1}{2}(\alpha-\theta)}. \tag{17}$$

Since

$$\int_0^1 \varphi d\eta = \tfrac{2}{3}, \quad \int_0^1 \varphi^2 d\eta = \tfrac{8}{15}$$

the equation (17) becomes after simplification

$$\frac{d\delta^2}{d\theta} + 9\delta^2 \cot \tfrac{1}{2}(\alpha-\theta) = \frac{60}{m}\nu a^2 \sin^2 \tfrac{1}{2}(\alpha-\theta) \tag{18}$$

We introduce the non-dimensional variable δ_1 given by

$$\delta_1 = \delta\left(\frac{2m}{\nu a^2}\right)^{\frac{1}{2}} \tag{19}$$

The equation (18) reduces to

$$\frac{d\delta_1^2}{d\theta} + 9\delta_1^2 \cot \tfrac{1}{2}(\alpha-\theta) = 120 \sin^2 \tfrac{1}{2}(\alpha-\theta) \tag{20}$$

or,
$$\frac{d}{d\theta}\left\{\delta_1^2/\sin^{18} \tfrac{1}{2}(\alpha-\theta)\right\} = \frac{120}{\sin^{16} \frac{1}{2}(\alpha-\theta)} \tag{21}$$

This equation when integrated under the condition $\delta_1 = 0$ when $\theta = 0$ gives

$$\delta_1^3 = 240 \sin^{18} \tfrac{1}{2}(\alpha-\theta)\{F[\cot \tfrac{1}{2}(\alpha-\theta)] - F[\cot \tfrac{1}{2}\alpha]\} \tag{22}$$

where

$$F(x) = x + \frac{7}{3}x^3 + \frac{21}{5}x^5 + 5x^7 + \frac{35}{9}x^9 + \frac{21}{11}x^{11} + \frac{7}{13}x^{13} + \frac{x^{15}}{15}\,. \tag{23}$$

This gives the thickness of the boundary layer as a function of θ.

Numerical Results. Taking the case of a channel where the liquid enters the channel at a point given by $\alpha = 30°$, values of δ_1 have been calculated for values of θ ranging from 0° to 25°. The results of the calculation are given in Table I and are shown graphically in Fig 2.

TABLE I

θ in degrees	1°	2°	3°	3° 30′	4°	5°	6°	7°	8°	9°	10°
δ_1	·319	.386	·407	·408	·406	·394	·377	·357	335	·313	·292
θ in degrees	12°	14°	15°	17°	19°	20°	21°	23°	25°		
δ_1	·249	·210	·190	·153	·119	·103	·088	·068	·036		

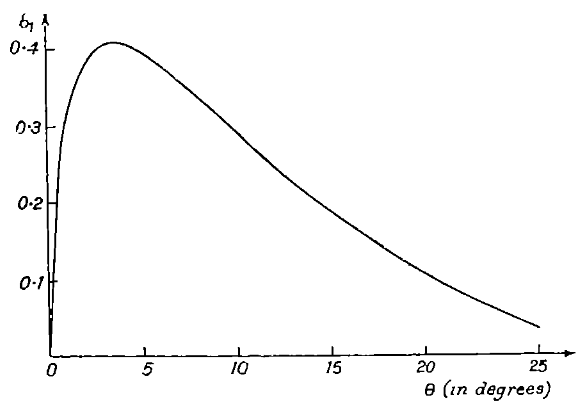

FIGURE 2

In conclusion, I wish to express my thanks to Dr. S. Ghosh for helpful suggestions and guidance.

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References

Goldstein, S. (1950), Modern Developments in Fluid Dynamics (Oxford), 101.
Pohlhausen, K. (1921), *Zeits. f. angew. Math und Mech* 1, 257

SOME RESULTS ON TOTAL INCLUSION FOR NÖRLUND SUMMABILITY

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(*Communicated by Dr. S. K Basu—Received January 11, 1954*)

Introduction. The necessary and sufficient conditions for the inclusion of the regular Nörlund method (N, p_n) corresponding to a given sequence (p_n) by the regular method (N, q_n) are wellknown [see Hardy, (1949) p. 67, Theorem 19]. In the present note the question of relative total inclusion* of a number of Nörlund methods for different sequences (p_n) and (q_n) has been considered.

1. Two methods of summability A and B are said to be *equivalent* if each includes the other, *i.e.* for any sequence (s_n), $A(s_n)\to l$ † implies $B(s_n)\to l$ and conversely. A and B are said to be *totally equivalent* if each includes the other *totally*, *i.e.*, $A(s_n)\to l$ (finite or infinite) implies $B(s_n)\to l$ and conversely.

The Nörlund method of summability (N, p_n) is defined by

$$N_n^p(s) = t_n = \frac{p_n s_0 + p_{n-1} s_1 + \ldots\ldots + p_0 s_n}{p_0 + p_1 + \ldots\ldots + p_n}$$

or,
$$N_n^p(s) = t_n = \frac{\sum\limits_{\nu=0}^{n} p_{n-\nu} s_\nu}{P_n} \tag{1}$$

where $p_n \geq 0$, $p_0 > 0$ and $P_n = p_0 + p_1 + p_2 + \ldots\ldots + p_n$.

(1) can also be written as

$$N_n^p(s) = t_n = \sum_{\nu=0}^{n} a_{n\nu} s_\nu \quad \text{where } a_{n\nu} = \frac{p_{n-\nu}}{p_n} \text{ if } \nu \leq n,$$

$$\text{and } a_{n\nu} = 0 \text{ if } \nu > n.$$

The Cesàro method of order $\alpha > 0$, denoted by (C, α) is a particular case of (N, p_n) where $p_n = \binom{n+\alpha-1}{\alpha-1}$.

Regarding the inclusion of one Norlund method by the other the following result is wellknown:

If (N, p_n) and (N, q_n) are regular, then, in order that (N, q_n) should include (N, p_n), it is necessary and sufficient that

$$|k_0|\,p_n + |k_1|\,p_{n-1} + \ldots\ldots + |k_{n-1}|\,p_1 + |k_n|\,p_0 \leq HQ_n. \tag{2}$$

* For definition see below. The quantities with which we are concerned are supposed to be real.

† Unless otherwise stated, l is finite.

where H is independent of n, and that

$$k_n/Q_n \to 0 \text{ as } n\to\infty, \quad (3)$$

k_n's being defined by

$$\sum_{n=0}^{\infty} k_n x^n = \frac{\sum_{n=0}^{\infty} q_n x^n}{\sum_{n=0}^{\infty} p_n x^n} = \frac{\sum_{n=0}^{\infty} Q_n x^n}{\sum_{n=0}^{\infty} P_n x^n} \quad (4)$$

where x is small.

We now prove the following theorem on total inclusion.

Theorem 1. *For the total inclusion of the (N, p_n) method by (N, q_n) in addition (2) and (3), it is necessary and sufficient that the k_n's as defined by (4) are all positive.*

Proof. If $N_n^q(s)$ denotes the Norlund means for (N, q_n), then we have the regular transformation

$$N_n^q(s) = \sum_{r=0}^{n} c_{nr} N_r^p(s) \quad \text{where } c_{nr} = \frac{k_{n-r}P_r}{Q_n} \text{ if } r\leq n, \text{ and } c_{nr}=0 \text{ if } r>n.$$

(see Hardy, *l.c.*, p. 67).

Therefore, for total regularity $c_{nr}\geq 0$ (see Hardy, p. 53, Theorem 10) except for atmost a finite set of values of r. Since P_n's and Q_n's are all positive, we must have

$$k_{n-r}\geq 0 \text{ for } r\geq\lambda, \text{ a fixed positive integer.}$$

Now taking $n=\lambda,\ \lambda+1,\ \lambda+2, \dots\ \dots\ \dots\ \dots$ it follows that

$$k_r\geq 0,\ r=0,\ 1,\ 2, \dots\ \dots\ \dots$$

Hence the result.

As an application of Theorem 1, we easily have the following theorem, the corresponding result for finite limits being wellknown (Hardy, *l.c.*, p. 67, Theorem 20).

Theorem 2. *If (N, q_n) is a regular Norlund method with $q_n\uparrow$, then $s_n\to\infty$ $(C, 1)$ implies $s_n\to\infty$ (N, q_n).*

Our next theorem is

Theorem 3. *Two regular Nörlund methods (N, p_n) and (N, q_n) cannot be totally equivalent unless they are identical.*

[For a parallel result involving Hansdroff summability see Basu (1949), p. 452].

Proof. Suppose that (N, p_n) and (N, q_n) are totally equivalent. Then as (N, q_n) totally includes (N, p_n), k_n's, as defined in (4) are all positive. Again, when (N, p_n) totally includes (N, q_n), l_n's as defined by

$$\sum_{n=0}^{\infty} l_n x^n = \frac{\sum_{n=0}^{\infty} p_n x^n}{\sum_{n=0}^{\infty} q_n x^n} = \frac{\sum_{n=0}^{\infty} P_n x^n}{\sum_{n=0}^{\infty} Q_n x^n} \quad (x \text{ small}) \quad (5)$$

are all positive.

From (4) and (5),

$$\sum_{n=0}^{\infty} k_n x^n \cdot \sum_{n=0}^{\infty} l_n x^n = 1$$

This being an identity, equating the coefficients of like powers of x, we have

$$k_o l_o = 1.$$

Therefore, $k_o \neq 0$ and $l_o \neq 0$.

Again, $k_o l_1 + k_1 l_o = 0$, but $k_o \neq 0$, $l_o \neq 0$ and k_n's and l_n's are all positive.

Therefore, $k_1 = l_1 = 0.$

Since $k_2 l_o + k_1 l_1 + k_o l_2 = 0$, therefore, $k_2 = l_2 = 0$, arguing as before.

Proceeding in this way, we have

$$k_3 = l_3 = 0 = k_4 = l_4 = 0, \ldots\ldots$$

Hence $k_n = l_n = 0$ for $n \geq 1$.

(4) therefore reduces to

$$k_o \sum_{n=0}^{\infty} p_n x^n = \sum_{n=0}^{\infty} q_n x^n$$

whence $k_o p_n = q_n.$

Thus when (N, p_n) and (N, q_n) are totally equivalent, (p_n) and (q_n) are connected by a relation $p_n = \lambda q_n$ for all n, λ being a positive constant; so that

$$N_n^p(s) = N_n^q(s).$$

Hence (N, p_n) and (N, q_n) are identical.

Some writers take for convenience $p_o = q_o = 1$, then $\lambda = 1$, and two sequences (p_n) are identical.

An interesting particular case of the Nörlund method is the method $(N, \cosh n^{\frac{1}{2}})$ which includes (C, α) for $\alpha > 0$. [Hill (1945), p. 94—98].

We shall prove the following Theorem in this connection.

Theorem 4. *$(N, \cosh n^{\frac{1}{2}})$ does not totally include (C, α) for $\alpha \geq 1{\cdot}55$.*

Proof. With the notations of Theorem **1**, here we take

$$p_n = \binom{n+\alpha-1}{\alpha-1} \text{ and } q_n = \cosh n^{\frac{1}{2}}.$$

Then as in (4), we have in this case

$$\sum_{n=0}^{\infty} k_n x^n \cdot \sum_{n=0}^{\infty} \binom{n+\alpha-1}{\alpha-1} x^n = \sum_{n=0}^{\infty} x^n \cosh n^{\frac{1}{2}},$$

where x is small.

This being an identity, equating the coefficients of like powers of x,

$$k_0\binom{\alpha}{\alpha-1}+k_1\binom{\alpha-1}{\alpha-1}=\cosh 1^{\frac{1}{2}}$$

or

$$k_1=\cosh 1-\alpha$$

As $\cosh 1<1\cdot55,$

Therefore, $k_1<1\cdot55-\alpha.$

Hence $k_1<0$ for $\alpha\geq 1\cdot55.$

Therefore, by Theorem 1. $(N, \cosh n^{\frac{1}{2}})$ does not totally include (C, α) for $\alpha\geq 1.55$. Hence the result.

The question whether $(N, \cosh n^{\frac{1}{2}})$ totally includes (C, α) for $0<\alpha<1\cdot55$ still remains open.

2. Given the series $\sum\limits_{n=0}^{\infty} a_n$ for which $s_n=\sum\limits_{\nu=0}^{n} a_\nu$, the *discontinuous Riesz means* of order $\alpha>0$ are given by

$$R_n^\alpha=\sum_{k=0}^{n-1}(1-k/n)^\alpha a_k, \quad \text{[Agnew (1938), p. 532]} \tag{6}$$

In case $R_n^\alpha\to l$, the $\sum\limits_{n=0}^{\infty} a_n$ is said to be summable (R, n, α) to l. It is wellknown that for $0<\alpha<1$, (C, α) and (R, n, α) are equivalent (Agnew, *l.c.*, p. 533): The question then naturally arises as to whether (R, n, α) totally includes (C, α) for $0<\alpha<1$ and *vice versa*. In this connection the following theorem will be proved.

Theorem 5. (R, n, α) does not totally include (C, α) f r $0<\alpha<1$.

Proof. (6) can be written as

$$R_n^\alpha=\sum_{k=0}^{n-1}\frac{1}{n^\alpha}\{(n-k)^\alpha-(n-k-1)^\alpha\}s_k,$$

so that (R, n, α) is the Nörlund method with $q_n=(n+1)^\alpha-n^\alpha$ and we have

$$R_n^\alpha=N_n^q(s)=\frac{\sum\limits_{k=0}^{n-1} q_{n-k-1}s_k}{Q_{n-1}} \text{ where } q_{n-k-1}=(n-k)^\alpha-(n-k-1)^\alpha$$

and $Q_{n-1}=q_0+q_1+\ldots\ldots+q_{n-1}$.

Now with the notations of (4),

$$\sum_{n=0}^{\infty} k_n x^n\cdot\sum_{n=0}^{\infty}\binom{n+\alpha-1}{\alpha-1}x^n=\sum_{n=0}^{\infty}\{(n+1)^\alpha-n^\alpha\}x^n,$$

where x is small.

This being an identity, equating the coefficients of like powers of x,

$$k_0\binom{\alpha}{\alpha-1}+k_1\binom{\alpha-1}{\alpha-1}=2^\alpha-1^\alpha$$

or,
$$\alpha+k_1=2^\alpha-1$$

whence
$$k_1=2^\alpha-1-\alpha.$$

Therefore, $k_1<0$, since $2^\alpha<1+\alpha$ for $0<\alpha<1$ [Hardy, Littlewood, Polya (1934), p. 40]. Hence by Theorem 1, (R, n, α) does not totally include $(C\ \alpha)$ for $0<\alpha<1$.

The question whether (C, α) totally includes (R, n, α) for $0<\alpha<1$ still remains open.

3. Let

$$t_n=\frac{\sum_{\nu=0}^{n} p_\nu s_\nu}{\log n} \qquad (n=0,1,2,\ldots\ldots) \tag{7}$$

where
$$p_\nu=1/(\nu+1).$$

In case $t_n\to l$, the series $\sum_{n=0}^{\infty} a_n$ is said to be summable by the *logarithmic mean method* L (Hardy, *l.c.*, p. 59).

Again, let

$$u_n=\frac{\sum_{\nu=0}^{n} p_\nu s_\nu}{P_n} \tag{8}$$

where $P_n=p_0+p_1+\ldots+p_n$ and p_ν's having the same significances as before.

In case $u_n\to l$, the series Σa_n is said to be summable by $\left(\overline{N}, \frac{1}{n+1}\right)$ *method* (see Hardy, *l.c.*, p. 59).

We shall now prove the following theorem.

Theorem 6. *The methods* $\left(\overline{N}, \frac{1}{n+1}\right)$ *and* L *are totally equivalent.*

Proof. From (7),

$$t_n \log n=\sum_{\nu=0}^{n} p_\nu s_\nu$$

and from (8),

$$u_n p_n=\sum_{\nu=0}^{n} p_\nu s_\nu$$

Thererfore,
$$t_n \log n=u_n p_n$$

or,
$$t_n=u_n\frac{n}{\log n}$$

Now $1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+1/n-\log n\to\gamma$, a constant,

so that
$$\frac{1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+1/n}{\log n}\to 1.$$

It follows from (9) that as $t_n\to l$ (finite or infinite) $u_n\to l$ and conversely (the case for l finite is aleady known).

Thus the theorem is proved.

It is known that each of the methods, the $\left(\overline{N}, \frac{1}{n+1}\right)$ method and the logarithmic mean method, is stronger than $(C, 1)$ for finite limits (Hardy, *l.c.*, p. 59). It is natural to ask whether this result is true even when the limit is infinite. Thus our last theorem is the following.

Theorem 7. *Both the methods, the* $\left(\overline{N}, \frac{1}{n+1}\right)$ *method and the logarithmic mean method, are stronger than* $(C, 1)$ *for infinite limits.*

Proof. In view of theorem 6, it is sufficient to consider either of the methods. We, therefore, take up the case for the $\left(\overline{N}, \frac{1}{n+1}\right)$ method.

From (8),
$$u_n = \sum_{\nu=0}^{n} a_{n\nu} s_\nu \text{ where } a_{n\nu} = \frac{p_\nu}{P_n} \text{ if } \nu \leq n,$$
$$\text{and } a_{n\nu} = 0 \text{ if } \nu > n.$$

As
$$C_n^1 = \frac{s_0+s_1+\ldots\ldots+s_n}{n+1}, \quad (n = 0, 1, 2, \ldots\ldots),$$

therefore
$$s_n = (n+1)C_n^1 - nC_{n-1}^1 \quad \text{taking} \quad C_{-1} = 0.$$

Now
$$u_n = \sum_{\nu=0}^{n} a_{n\nu}\{(\nu+1)C_\nu^1 - \nu C_{\nu-1}^1\}$$
$$= \sum_{\nu=0}^{n} (\nu+1) C_\nu^1 (a_{n\nu} - a_{n,\nu+1})$$

or,
$$u_n = \sum_{\nu=0}^{n} b_{n\nu} C_\nu^1 \tag{10}$$

where
$$b_{n\nu} = (\nu+1) \quad (a_{n\nu} - a_{n,\nu+1})$$

The transformation (10) is regular. For total regularity we must have $b_{n\nu} \geq 0$ for $\nu \geq r$, a fixed positive integer.

Now $b_{n\nu} = (\nu+1)\ (a_{n\nu} - a_{n,\nu+1}) \geq 0$ for all ν.

Since $$(a_{n\nu} - a_{n,\,\nu+1}) \geq 0 \text{ for all } \nu.$$

Hence the $\left(\overline{N}, \dfrac{1}{n+1}\right)$ method totally includes $(C, 1)$. This proves the theorem.

Finally, I am grateful to Dr. S. K. Basu for his valuable suggestion and help in the preparation of this note.

References.

Agnew, R. P. (1933), *Trans. American Math. Soc.*, **35**, p. 532.
Basu, S. K (1949), *Proc. Lond. Math. Soc.* (2), **50**, 447.
Hardy, G. H. (1949), *Divergent Series* Oxford.
Hardy, G H., Littlewood J E., Polya, G. (1934), *Inequalities*, Cambridge.
Hill J. D. (1945), *Amer. Journal of Math*, **67**.

AXISYMMETRIC FLOW IN PERFECT FLUID—I

MOTION ABOUT A SPHERIOD AND CIRCULAR DISC.

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(*Communicated by Prof B R Seth—Received November 19, 1953*)

1. Introduction. Taylor (1922) has discussed the motion of a perfect fluid past a sphere, held at rest, when the fluid at infinity rotates with a uniform angular velocity Ω_0 about an axis through the centre of the sphere and moves with a velocity U_0 parallel to the axis. Nigam (1953) has shown that the problem can also be worked out when the sphere is replaced by any other body of revolution. In this paper we have worked out this problem for a spheroid and deduced the flow for a circular disc as a limiting case of an oblate spheroid.

The equations governing the motion of perfect incompressible fluid are (Goldstein, 1938)

$$\frac{\partial \mathbf{V}}{\partial t} - \mathbf{V}\times\boldsymbol{\omega} = -\operatorname{grad}\left(P/\varrho + F + \frac{\mathbf{V}^2}{2}\right) \tag{1}$$

$$\operatorname{div} \mathbf{V} = 0 \tag{2}$$

In what follows α, β denote the general orthogonal curvilinear coordinates in the meridian plane and γ the azimuthal angle; h_1, h_2, h_3 denote the elements of lengths in the directions of α, β, γ increasing respectively, h_3 in addition represents the distance of any point from the axis of rotation. u, v, w denote the components of velocity vector $\mathbf{V}$ in the direction of α, β, γ increasing and ξ', η', ζ' are the components of the vorticity vector $\boldsymbol{\omega}$. The motion being symmetrical about an axis all quantities are independent of the azimuthal angle γ.

Equation (2) can be written as

$$\frac{\partial}{\partial\alpha}(h_2\, h_3\, u) + \frac{\partial}{\partial\beta}(h_1\, h_3\, v) = 0 \tag{3}$$

We introduce Stokes' stream function ψ such that

$$u = \frac{1}{h_2 h_3}\frac{\partial\psi}{\partial\beta},\ v = -\frac{1}{h_1 h_3}\ \frac{\partial\psi}{\partial\alpha},\ w = \frac{\Omega'}{h_3} \tag{4}$$

With these values of u, v, w, components of vorticity are given by

$$\xi' = \frac{1}{h_2 h_3}\ \frac{\partial\Omega'}{\partial\beta},\ \eta' = -\frac{1}{h_1 h_3}\ \frac{\partial\Omega'}{\partial\alpha},\ \zeta' = -\frac{1}{h_3}D^2\psi \tag{5}$$

where
$$D^2 = \frac{h_3}{h_1 h_2}\left[\frac{1}{\partial\alpha}\left(\frac{h_2}{h_1 h_3}\ \frac{\partial}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{h_1}{h_2 h_3}\ \frac{\partial}{\partial\beta}\right)\right] \tag{6}$$

The third component of the vector equation of vorticity is

$$\frac{\partial}{\partial t}(D^2\psi)+\frac{2\Omega'}{h_1h_2h_3^2}\frac{\partial(\Omega',h_3)}{\partial(\alpha,\beta)}-\frac{1}{h_1h_2h_1}\frac{\partial(\psi,D^2\psi)}{\partial(\alpha,\beta)}+\frac{2D^2\psi}{h_1h_2h_3^2}\frac{\partial(\psi,h_3)}{\partial(\alpha.\beta)}=0 \tag{7}$$

The third component of the vector equation of motion is

$$\frac{\partial\Omega'}{\partial t}-\frac{1}{h_1h_2h_3}\frac{\partial(\psi,\Omega')}{\partial(\alpha,\beta)}=0 \tag{8}$$

For any body of revolution the above equations can be simplified in the following manner (Nigam, 1953). If

$$h_3w=\Omega'=K\psi,\ [\psi_\infty]=h_3^2\Omega_0/K \tag{9}$$

Equation (8) is satisfied and (7) reduces to

$$\frac{D}{Dt}\left(\frac{\zeta'}{h_3}-\frac{K^2\psi}{h_3^2}\right)=0 \tag{10}$$

or

$$\frac{\zeta'}{h_3}-\frac{K^2\psi}{h_3^2}=H(\psi) \tag{11}$$

Substituting for ζ', the equation for ψ is

$$(D^2+K^2)\psi=+K\Omega_0h_3^2 \tag{12}$$

whence

$$\psi=+\frac{\Omega_0}{K}h_3^2+A_1\psi_1+A_2\psi_2 \tag{13}$$

where ψ_1 and ψ_2 are the appropriate solutions of

$$(D^2+K^2)\psi=0 \tag{14}$$

For a sphere Taylor gets

$$\psi=\frac{U_0}{2}\varrho^2\sin^2\theta+A\left[\cos(K\varrho+\varepsilon)-\frac{\sin(k\varrho+\varepsilon)}{K\varrho}\right],\ K=\frac{2\Omega_0}{U_0} \tag{15}$$

As perfect fluid slips along the boundary of the solid, Taylor imposes only one boundary condition, namely that the normal velocity is zero on $r=a$. This also makes $w=0$ on $r=a$ and leaves the problem indeterminate. Taylor thinks that this indeterminacy arises because of the possibility of starting the motion in an infinite number of ways.

Taylor has further shown that it is possible to find out solutions in which $u=v=w=0$ on $r=a$. These would correspond to the motion in which there is no slipping between the fluid and the surface of the sphere. In the ordinary irrotational motion of a perfect fluid past a solid body it is not possible to satisfy the condition of the zero slip on the surface of the body. This assumption of slipping between the surface of the solid body and a perfect fluid vitiates all the hydrodynamical theories of the motion of solids in real fluids. Taylor is therefore inclined to think that the solutions which represent the motion of fluid rotating and moving uniformly at infinity past a sphere and

give zero slip on its surface might approximate more closely to reality than the ordinary irrotational solutions representing the motion of fluid past a sphere in which we allow for slip.

As in the case of the sphere we get an infinite number of possible motions round the spheroid held at rest in a stream of liquid uniformly rotating and moving at infinity. We further show that it is possible to satisfy the condition $u = v = w = 0$ on the surface, and the condition of uniform flow and rotation at infinity. The notation used by Stratton-Morse-Chu-Hunter (1941) is used throughout this paper.

2. Prolate Spheroid. For a prolate spheroid we introduce the system of orthogonal coordinates defined by

$$z + ir = C \cosh(\alpha + i\beta)\,; \; \varphi = \gamma \tag{16}$$

where r, z, are the cylindrical coordinates in the meridian plane and γ is the azimuth.

We then have $h_1 = h_2 = C(\cosh^2\alpha - \cos^2\beta)^{\frac{1}{2}}$, $h_3 = C \sinh\alpha \sin\beta$ and

$$D^2 = \frac{1}{C^2(\cosh^2\alpha - \cos^2\beta)}\left[\frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2} - \coth\alpha\frac{\partial}{\partial\alpha} - \cot\beta\frac{\partial}{\partial\beta}\right].$$

Equation (14) then yields

$$\left[\frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2} - \coth\alpha\frac{\partial}{\partial\alpha} - \cot\beta\frac{\partial}{\partial\beta} + K^2C^2(\cosh^2\alpha - \cos^2\beta)\right]\psi = 0. \tag{17}$$

Putting $\qquad \psi = C \sinh\alpha \sin\beta\, S(\beta) R(\alpha)\,; \; \cosh\alpha = \xi, \; \cos\beta = \eta$

equation (17) separates into the two following equations.

$$\frac{d}{d\eta}(\eta^2 - 1)\frac{dS}{d\eta} + \left\{\lambda + K^2C^2\eta^2 - \frac{1}{\eta^2 - 1}\right\} S = 0 \tag{18}$$

$$\frac{d}{d\xi}(\xi^2 - 1)\frac{dR}{d\xi} + \left\{\lambda + K^2C^2\xi^2 - \frac{1}{\xi^2 - 1}\right\} R = 0 \tag{19}$$

where λ is the constant of separation. Equation (18) and (19) are identical with the equation (K) and (L) (Stretton *et al.*, *l.c.*, p. 62) which arise in the discussion of the prolate spheroidal functions. Equations (K) and (L) contain a third paramater m and have solutions for suitably related values of the parameter in series of Associated Legendre's Polynomials or Bessel functions of half order. In our case m has value unity. Thus for various characteristic values of λ there are 'angular solutions' $S_{1l}^{(1)}$ of (18) which are finite throughout the range $-1 \leq \eta \leq 1$ and are expressed as infinite series of Associated Legendre's polynomials in the form

$$S_{1l}^{(1)}(KC, \eta) = \sum_{n=0,1}^{\infty}{}' d_n^l \, P_{n+1}^{(1)}(\eta) \tag{20}$$

where the prime indicates summation over even or odd values of n, according as l is even or odd. For radial solutions which are denoted by $R_{1l}^{(1)}(KC\xi)$ and $R_{1l}^{(2)}(KC\xi)$ where

$$R_{1l}^{(1)}(CK\xi) = \frac{(\xi^2-1)\left(\frac{\pi}{2KC\xi^3}\right)^{\frac{1}{2}} \sum_{n=0,1}^{\infty}{}' \, i^{l-n}\, d_n^l \, (n+1)(n+2)\, J_{n+3/2}(KC\xi)}{\sum_{n=0,1}^{\infty}{}' \, d_n^l \, (n+1)(n+2)} \tag{21}$$

$$\xrightarrow[KC\xi\to\infty]{} \frac{1}{KC\xi}\sin\left(KC\xi - \frac{l+1}{2}\pi\right)$$

$$R_{1l}^{(2)}(KC\xi) = \frac{(\xi^2-1)^{\frac{1}{2}}\left(\frac{\pi}{2KC\xi^3}\right)^{\frac{1}{2}} \sum_{n=0,1}^{\infty}{}' \, i^{l+n+4} d_n^l (n+1)(n+2)\, J_{-n-3/2}(KC\xi)}{\sum_{n=0,1}^{\infty}{}' \, d_n^l (n+1)(n+2)} \tag{22}$$

$$\xrightarrow[KC\xi\to\infty]{} -\frac{1}{KC\xi}\cos\left(KC\xi - \frac{l+1}{2}\pi\right)$$

The summation of these radial solutions are from $n = 0$ or 1 to $n = \infty$. The complete solution of the equation (14) is therefore

$$\psi = +\frac{\Omega_0}{K}C^2 \sinh^2\alpha \sin^2\beta + C\sinh\alpha\sin\beta \sum_{l=0}^{\infty}[A_l R_{1l}^{(1)} + B_l R_{1l}^{(2)}]S_{1l}^{(1)} \tag{23}$$

3. **Expressions for the velocity components.** On the surface of the spheroid where $\alpha = \alpha_0$ we have

$$u = \frac{1}{h_2 h_3}\frac{\partial\psi}{\partial\beta} = 0 \quad \text{or} \quad \frac{\partial\psi}{\partial\beta} = 0 \tag{24}$$

$$w = \frac{K\psi}{h_3} = 0 \qquad \text{or} \quad \psi = 0 \tag{25}$$

The condition $\psi = 0$ on $\alpha = \alpha_0$; makes the spheroid a streamline. This therefore satisfies the condition of zero normal velocity $\left(\frac{\partial\psi}{\partial\beta} = 0\right)$ on the boundary. If $+U_0$ is the velocity of the fluid at infinity we find

$$\left[+\frac{CU_0\sinh\alpha\cos\beta}{h_1}\right]_{\alpha=\infty} = \left[\frac{1}{h_1 h_3}\left\{+\frac{2\Omega_0}{K}C^2\sinh\alpha\sin\beta\cos\beta\right.\right.$$

$$\left.\left. + C\cos\beta\sinh\alpha\sum_{l=0}^{\infty}(A_l R_{1l}^{(1)} + B_l R_{1l}^{(2)})S_{1l}^{(1)} + C\sin\beta\sinh\alpha\sum(A_l R_{1l}^{(1)} B_l R_{1l}^{(1)})\frac{\partial}{\partial\beta}S_{1l}^{(1)}\right\}\right]_{\alpha=\infty} \tag{27}$$

Putting $\alpha = \infty$ equation (26) gives $K = \dfrac{2\Omega_0}{U_0}$

Making use of the condition (25) we get

$$+\frac{\Omega_0}{K} C \sinh\alpha_0 \sin\beta = \sum_{l=0}^{\infty}\left(A_l R_{1l}^{(1)} + B_l R_{1l}^{(2)}\right) S_{1l}^{(1)}$$

Multiplying both sides by $S_{1l}^{(1)}$ and integrating with respect to β between 0 to π, from the orthogonal property of $S_{1l}^{(1)}$ we have

$$A_l R_{1l}^{(1)} + B_l R_{1l}^{(1)}\Big|_{\alpha=\alpha_0} = \frac{\Omega_0}{K} C \sinh\alpha_0 \, T_l' \tag{27}$$

where T_l' is zero for odd values of l and has the value

$$(4/3) d_0^l \Big/ {\sum_{n=0}^{\infty}}' \frac{2(d_n^l)^2 (n+1)(n+2)}{(2n+3)} \tag{28}$$

when l is even Any values of A_l and B_l which satisfy (27) lead to the possible solutions of the problem. However if we further impose the condition of no slip on the surface of the spheroid both the constants become determinate and give a unique solution. This makes $\frac{\partial\psi}{\partial\alpha} = 0$ on $\alpha = \alpha_0$. Making use of this condition we get

$$\frac{2\Omega_0}{K} C \sinh\alpha_0 \cosh\alpha_0 \sin\beta = \sum_{l=0}^{\infty}\Big[A_l\Big(\cosh\alpha_0 \, R_{1l}^{(1)}\Big|_{\alpha=\alpha_0} + \sinh\alpha_0 \, R'^{(1)}_{1l}\Big|_{\alpha=\alpha_0}\Big)$$
$$+ B_l\Big(\cosh\alpha_0 R_{1l}^{(1)}\Big|_{\alpha=\alpha_0} + \sinh\alpha_0 R'^{(1)}_{1l}\Big|_{\alpha=\alpha_0}\Big)\Big] S_{1l}^{(1)}$$

where a dash denotes differentiation with respect to α. Multiplying both sides by $S_{1l}^{(1)}$ and integrating between the limits 0 to π, with respect to β and making use of the orthogonal property of $S_{1l}^{(1)}$ we have

$$A_l\Big[\cosh\alpha_0 \, R_{1l}^{(1)}\Big|_{\alpha=\alpha_0} + \sinh\alpha_0 \, R'^{(1)}_{1l}\Big|_{\alpha=\alpha_0}\Big] + B_l\Big[\cosh\alpha_0 \, R_{1l}^{(2)}\Big|_{\alpha=\alpha_0} + \sinh\alpha_0 \, R'^{(2)}_{1l}\Big|_{\alpha=\alpha_0}\Big]$$
$$= \frac{2\Omega_0}{K} C \cosh\alpha_0 \sinh\alpha_0 \, T_l' \tag{29}$$

where T_l' is zero when l is odd and has the value given by (28) when l is even. From equation (27) and (29) it therefore follows that A_l, B_l vanish for odd values of l and the infinite series of radial solutions contains terms having only even suffixes in l. Solving equations (27) and (29) for even values of l, we have

$$A_l = \frac{-\frac{\Omega_0}{K} C T_l' \left\{\sinh\alpha_0 \, R'^{(2)}_{1l}\Big|_{\alpha=\alpha_0} - \cosh\alpha_0 \, R_{1l}^{(2)}\Big|_{\alpha=\alpha_0}\right\}}{R_{1l}^{(2)}\Big|_{\alpha=\alpha_0} R'^{(1)}_{1l}\Big|_{\alpha=\alpha_0} - R_{1l}^{(1)}\Big|_{\alpha=\alpha_0} R'^{(2)}_{1l}\Big|_{\alpha=\alpha_0}} \tag{30}$$

$$B_l = \frac{-\frac{\Omega_0}{K} C T_l' \left\{\cosh\alpha_0 \, R_{1l}^{(1)} - \sinh\alpha_0 \, R'^{(1)}_{1l}\right\}}{R_{1l}^{(2)}\Big|_{\alpha=\alpha_0} R'^{(1)}_{1l}\Big|_{\alpha=\alpha_0} - R_{1l}^{(1)}\Big|_{\alpha=\alpha_0} R'^{(2)}_{1l}\Big|_{\alpha=\alpha_0}} \tag{31}$$

Using the values of A_l, B_l, the expressions for the velocity components can be found from (23) and (4). As a limiting case when $\alpha_0 = 0$ making use of the special values of the prolate spheroidal functions given in (Stratton *et al.*, *l.c.*, p. 68) we have from the equations (27) and (29)

$$B_l = 0$$

$$A_l = \frac{6\Omega_0 T_l'(\frac{1}{2}l+\frac{1}{2})!}{K^2\pi^{\frac{1}{2}}d_0{}^l(l/2)!}$$

and the stream function becomes

$$\psi = C^2 \sinh^2\alpha \sin^2\beta + C \sinh\alpha \sin\beta \sum_{l=0}^{\infty} \frac{6\Omega_0}{K^2} \frac{T_l'}{d_0{}^l} \frac{(\frac{1}{2}l+\frac{1}{2})!}{(l/2)!} R_{1l}^{(1)} S_{1l}^{(1)}$$

This represents the flow due to a fluid rotating and moving at infinity past a pressure line stretched between the points $-C$ to C along the axis of the rotating fluid. In the limiting case when $C = 0$ this corresponds to the flow of fluid rotating and moving at infinity past a pressure point of its axis. Then becomes $\psi = \frac{1}{2}U_0 \varrho^2 \sin^2\theta + \sin^2\theta\varrho^{\frac{1}{2}} \frac{6\Omega_0}{K^{\frac{3}{2}}} J_{3/2}(K\varrho)$. This agrees with the r sult found by Taylor.

4. Oblate spheroid. For an oblate spheroid we introduce the system of coordinates defined by

$$z + ir = C \sinh(\alpha + i\beta), \quad \varphi = \gamma \tag{33}$$

We then get $h_1 = h_2 = C(\sinh^2\alpha + \cos^2\beta)^{\frac{1}{2}}$, $h_3 = C \cosh\alpha \sin\beta$

$$D^2 = \frac{1}{C^2(\sinh^2\alpha + \cos^2\beta)} \left[\frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2} - \tanh\alpha\frac{\partial}{\partial\alpha} - \cot\beta \frac{\partial}{\partial\beta} \right]$$

Equation (14) then yields

$$\left[\frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2} - \tanh\alpha\frac{\partial}{\partial\alpha} - \cot\beta\frac{\partial}{\partial\beta} + K^2C^2(\sinh^2\alpha + \cos^2\beta) \right] \psi = 0 \tag{34}$$

Putting $\psi = C \cosh\alpha \sin\beta\, R_{(\alpha)}S_{(\beta)}$, $\sinh\alpha = \xi$, $\cos\beta = \eta$ (35)

equation (34) separates into the two following equations

$$\frac{d}{d\eta}(\eta^2 - 1)\frac{dS}{d\eta} + \left[\lambda - K^2C^2\eta^2 - \frac{1}{\eta^2 - 1}\right] S = 0 \tag{36}$$

$$\frac{d}{d\xi}(\xi^2 - 1) \frac{dR}{d\xi} + \left[\lambda + K^2C^2\xi^2 + \frac{1}{\xi^2 + 1}\right] R = 0 \tag{37}$$

where λ is the constant of separation. Equation (36) can be obtained from (18) by changing KC to iKC. The equation (37) can be obtained from (19) by changing KC to iKC and ξ to $-i\xi$. Therefore the solutions of the equations (36) and (37) can be obtained from those of the equations (18), (19) by changing KC to $-iKC$ and ξ to $i\xi$. Thus the angular solution is given by $S_{11}^{(1)}(-iKC, \eta) = \sum_{n=0,1}^{\infty}{}' f_n^1 P_{n+1}^{(1)}(\eta)$ and the radial solution is

$$R_{1l}^{(1)}(-iKC, i\xi) = \frac{(1+\xi^2)^{\frac{1}{2}}\left(\frac{\pi}{2KC\xi^3}\right)^{\frac{1}{2}} \sum\limits_{n=0,1}^{\infty} i^{l-n} f_n^l (n+1)(n+2) J_{n+3/2}(KC\xi)}{\sum\limits_{n=0,1}^{\infty}{}' f_n^l (n+1)(n+2}$$

$$\xrightarrow[KC\xi\to\infty]{} \frac{1}{KC\xi} \sin\left(KC\xi - \frac{l+1}{2}\pi\right) \tag{38}$$

$$R_{1l}^{(2)}(iKC, i\xi) = \frac{(1+\xi^2)^{\frac{1}{2}}\left(\frac{\pi}{2KC\xi^3}\right)^{\frac{1}{2}} \sum\limits_{n=0,1}^{\infty}{}' i^{l+n+1} f_n^l (n+1)(n+2) J_{-n-3/2}(KC\xi)}{\sum\limits_{n=0}^{\infty}{}' f_n^l (n+1)(n+2)}$$

$$\xrightarrow[KC\xi\to\infty]{} -\frac{1}{KC\xi} \cos\left(KC\xi - \frac{l+1}{2}\pi\right)$$

$$\xi = \sinh\alpha \tag{39}$$

The complete solution of (14) is

$$\psi = \frac{\Omega_0}{K} C^2 \cosh^2\alpha \sin^2\beta + C\cosh\alpha \sin\beta \sum_{l=0}^{\infty}\left[A_l R_{1l}^{(1)} + B_l R_{1l}^{(2)}\right] S_{1l}^{(1)} \tag{40}$$

5. Expressions for the velocity components. If $-U_0$ is the velocity of the fluid at infinity, we find

$$\left[\frac{C U_0 \cosh\alpha \cos\beta}{h_1}\right]_{\alpha=\infty} = \left[\frac{1}{h_1 h_3}\left\{\frac{2\Omega_0}{K} C^2 \cosh^2\alpha \sin\beta\cos\beta\right.\right.$$

$$+ C\cosh\alpha\cos\beta \sum_{l=0}^{\infty}\left[A_l R_{1l}^{(1)} + B_l R_{1l}^{(2)}\right] S_{1l}^{(1)}$$

$$\left.\left. + C\cosh\alpha\sin\beta \sum_{l=0}^{\infty}\left[A_l R_{1l}^{(1)} + B_l R_{1l}^{(1)}\right]\frac{\partial}{\partial\beta} S_{1l}^{(1)}\right\}\right]_{\alpha=\infty}$$

Putting $\alpha = \infty$ we get $\qquad K = \dfrac{2\Omega_0}{U_0}$.

From the condition (25) we have

$$\frac{\Omega_0}{K} C\cosh\alpha_0 \sin\beta = \sum_{l=0}^{\infty}\left(A_l R_{1l}^{(1)}\Big|_{\alpha=\alpha_0} + B_l R_{1l}^{(1)}\Big|_{\alpha=\alpha_0}\right) S_{1l}^{(1)} \tag{41}$$

Multiplying both sides again by $S_{1l}^{(1)}$ and integrating between the limits 0 to π with respect to β and using the orthogonal property of $S_{1l}^{(1)}$ we get

$$\frac{\Omega_0}{K} C\cosh\alpha_0 T_l' = A_l R_{1l}^{(1)}\Big|_{\alpha=\alpha_0} + B_l R_{1l}^{(2)}\Big|_{\alpha=\alpha_0} \tag{42}$$

where T_l' vanishes when l is odd and has the value

$$(4/3)f_0^l \Big/ \sum_{n=0}^{\infty} \frac{2(f_n^l)^2(n+1)(n+2)}{(2n+3)} \tag{43}$$

when n is even. Any values of A_l, B_l which satisfy (42) lead to the possible solutions of the problem. If we further impose the condition of no slip on the surface $\alpha = \alpha_0$ the constants become determinate and give a unique solution. This makes $\partial\psi/\partial\alpha = 0$ on $\alpha = \alpha_0$. Making use of this condition we have

$$\frac{2\Omega_0}{K} C \cosh\alpha_0 \sinh\alpha_0 \sin\beta = \sum_{l=0}^{\infty}\Big[A_l (\sinh\alpha_0 \underset{\alpha=\alpha_0}{R_{1l}^{(1)}} + \cosh\alpha_0 \underset{\alpha=\alpha_0}{R_{1l}^{\prime(1)}}) +$$

$$B_l(\sinh\alpha_0 \underset{\alpha=\alpha_0}{R_{1l}^{(2)}} + \cosh\alpha_0 \underset{\alpha=\alpha_0}{R_{1l}^{\prime(2)}})\Big] S_{1l}^{(1)}$$

where a dash denotes differentiation with respect to α. Multiplying both sides by $S_{1l}^{(1)}$ and integrating between the limits 0 to π with respect to β and using the orthogonal property of $S_{1l}^{(1)}$ we have

$$\frac{2\Omega_0}{R} C \cosh\alpha_0 \sinh\alpha_0 T_l' = A_l (\sinh\alpha_0 \underset{\alpha=\alpha_0}{R_{1l}^{(1)}} + \cosh\alpha_0 \underset{\alpha=\alpha_0}{R_{1l}^{(1)}}$$

$$+ B_l (\sinh\alpha_0 \underset{\alpha=\alpha_0}{R_{1l}^{(2)}} + \cosh\alpha_0 \underset{\alpha=\alpha_0}{R_{1l}^{\prime(2)}}) \tag{44}$$

Solving equations (42) and (44) for even values of

$$A_l = \frac{-\frac{\Omega_0}{K} C T_l' \Big[\sinh\alpha_0 \underset{\alpha=\alpha_0}{R_{1l}^{(2)}} - \cosh\alpha_0 \underset{\alpha=\alpha_0}{R_{1l}^{\prime(2)}}\Big]}{\underset{\alpha=\alpha_0}{R_{1l}^{(1)}} \underset{\alpha=\alpha_0}{R_{1l}^{\prime(2)}} - \underset{\alpha=\alpha_0}{R_{1l}^{(2)}} \underset{\alpha=\alpha_0}{R_{1l}^{\prime(1)}}} \tag{45}$$

$$B_l = \frac{\frac{\Omega_0}{K} C T_l' \Big[\sinh\alpha_0 \underset{\alpha=\alpha_0}{R_{1l}^{(1)}} - \cosh\alpha_0 \underset{\alpha=\alpha_0}{R_{1l}^{\prime(1)}}\Big]}{\underset{\alpha=\alpha_0}{R_{1l}^{(1)}} \underset{\alpha=\alpha_0}{R_{1l}^{\prime(2)}} - \underset{\alpha=\alpha_0}{R_{1l}^{(2)}} \underset{\alpha=\alpha_0}{R_{1l}^{\prime(1)}}} \tag{46}$$

Using these values of A_l and B_l the expressions for the velocity components can be found from (40) and (4).

6. Circular Disc. Circular disc is the limiting form of an oblate spheroid for which ξ_0 in zero. Putting $\xi_0 = \sinh\alpha_0 = 0$ in equations (43) and (44) and making use of the special values given in (Stratton *et al.*, *l.c.*, p. 70-71) we get $B_l = 0$

$$A_l = \frac{\Omega_0}{K} C T_l' / R_{1l}^{(1)} (-iKC, 0) \tag{47}$$

where
$$R_{1l}^{(1)} (-iKC, 0) = \frac{i^l \pi^{\frac{1}{2}} CK f_0^l}{2(3/2)! \sum_{n=0}^{\infty} f_n^l (n+1)(n+2)} \tag{48}$$

It is interesting to note that in the case of the circular disc only one set of constants satisfies the conditions $u = w = 0$ and also the condition $v = 0$ on $\xi_0 = 0_1$

The stream function is given by

$$\psi = \frac{\Omega_0}{K} C^2 \cosh^2\alpha \sin^2\beta + C \cosh\alpha \sin\beta \sum_{l=0}^{\infty} \frac{+\Omega_0 C T_l'}{K R_{1l}^{(1)}(\imath KC, 0)} R_{1l}^{(1)} S_{1l}^{(1)} \tag{49}$$

where

$$k = 2\Omega_0 / U_0$$

$$R_{1l}^{(1)} = \frac{(\xi^2+1)^{\frac{1}{2}} \left(\frac{\pi}{2KC\xi^3}\right)^{\frac{1}{2}} \sum_{n=0}^{\infty} i^{l-n} f_n^l (n+1)(n+2) J_{n+3/2}(KC\xi)}{\sum_{n=0}^{\infty}{}' (n+1)(n+2)}$$

$$\xi = \sinh\alpha$$

and C is the radius of the disc.

7. **Conclusions** · Proceeding to the limit when $C = 0$ and $\xi_0 = \infty$ such that $C\xi = \varrho$ it is readily observed that $S_{1l}^{(1)}(KC, \cos\beta) \rightarrow P_{1l}^{(1)}(\cos\theta)$

$$A_l R_{1l}^{(1)}(KC\xi) \rightarrow A_l \varrho^{\frac{1}{2}} J_{l+3/2}(K\varrho)$$

$$B_l R_{1l}^{(1)}(KC\xi) \rightarrow B_l \varrho^{\frac{1}{2}} J_{-l-3/2}(K\varrho)$$

If the conditions $u = v = w = 0$ are now satisfied all A_l' and B_l' vanish except when l is zero. In this case

$$A_0' = \frac{-\frac{U_0}{4} a^{\frac{1}{2}} [2aJ_{-3/2}(Ka) - 3J_{-3/2}(Ka)]}{J_{3/2}(Ka)J'_{-3/2}(Ka) - J_{-3/2}(Ka)J'_{3/2}(Ka)} \tag{50}$$

$$B_0' = \frac{-\frac{U_0}{4} a^{\frac{1}{2}} [3J_{3/2}(Ka) - 2a\, J'_{-3/2}(Ka)}{J_{3/2}(Ka)J'_{-3/2}(Ka) - J_{-3/2}(Ka)J'_{3/2}(Ka)} \tag{51}$$

and the stream function is given by

$$\psi = \frac{U_0}{K} \varrho^2 \sin^2\theta + \varrho^{\frac{1}{2}} \sin^2\theta [A'_0 J_{3/2}(Ka) + B'_0 J_{-3/2}(Ka)] \tag{52}$$

This is Taylor's solution of the flow of fluid rotating and moving at infinity past a sphere held at rest within it. Further putting $C = 0$ and $\xi = \infty$ such that $C\xi = \varrho$ in

$$\psi = \frac{\Omega_0}{K} C^2 \sinh^2\alpha \sin^2\beta + C \sinh\alpha \sin\beta \sum_{l=0}^{\infty} [A_l R_{1l}^{(1)} + B_l R_{1l}^{(2)}] S_{1l}^{(1)} \tag{53}$$

it reduces to

$$\psi = \frac{\Omega_0}{K} \varrho^2 \sin^2\theta + \varrho^{\frac{1}{2}} \sin^2\theta \sum_{l=0}^{\infty} (A_l' J_{l+3/2}(K\varrho) + B_l' J_{-l-3/2}(K\varrho)) P_{l+1}^{(1)} \cos\theta \tag{54}$$

where ϱ, θ, φ are the spherical polar coordinates of any point. Equation (54) is Long's generalisation of Taylor's solution.

I thank Dr. S. D. Nigam for suggesting this problem to me and for his kind interest in the preparation of this paper.

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References

Taylor, G. I., (1922), *Proc. Roy. Soc.*, **102**, 180.
Nigam, S. D., (1953), Unpublished. *Slip in perfect fluids.*
Goldstein, (1936), *Modern Development in Fluid Dynamics*
Stratton-Morse-Chu-Hunter, (1941), *Elliptic Cylinder and Spheroidal Wave Functions*, London.
Long, R. R., (1952), *Technical Report* No. 1., The John Hopkins University.

STRESS DISTRIBUTIONS IN A THIN PLATE AROUND A HOLE IN THE FORM OF A LOOP OF LEMNISCATE OF BERNOULLI

By

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(*Received—March 15, 1954.*)

Introduction. Following a method developed by Sen (1948), problems of stress distributions in a thin plate around a hole in the form of a loop of Lemniscate of Bernoulli have been solved in this paper, it being assumed that the plate is under either a uniform shear 'S' or a uniform tension T parallel to the x-axis at large distances from the hole. The method consists of expressing the stresses explicitly in terms of some plane harmonic functions, such that the proper choice of any one of them leads to the complete solution of the problem.

1. Form of the boundary and the expressions for the stress components. Let us consider the transformation given by

$$z^2 = c^2(1+e^{2\zeta}) \tag{1.1}$$

where c is a real constant, $z = x+iy$ and $\zeta = \xi+i\eta$.

It can be easily verified that the curve $\xi = 0$ is a lemniscate of Bernoulli of which the polar equation is

$$r^2 = 2c^2 \cos 2\theta.$$

We shall consider the boundary of the hole to be a loop of the lemniscate which encloses the region

$$-\infty \leq \xi \leq 0 \text{ and } -\tfrac{1}{2}\pi \leq \eta \leq +\tfrac{1}{2}\pi.$$

In this case, we have

$$\frac{1}{h^2} = \left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2 = \frac{c^2 e^{4\xi}}{(1+e^{4\xi}+2e^{2\xi}\cos 2\eta)^{\frac{1}{2}}} \tag{1.2}$$

and

$$r^2 = x^2+y^2 = c^2(1+e^{4\xi}+2e^{2\xi}\cos 2\eta)^{\frac{1}{2}} \tag{1.3}$$

Assuming the plate to be in a state of generalized plane stress, we have, in the absence of body forces, the mean stresses $\widehat{\xi\xi}$, $\widehat{\eta\eta}$ and $\widehat{\xi\eta}$ satisfying the equations of equilibrium and compatibility as given by [*cf.* Sen, 1948]

$$\frac{8\widehat{\xi\xi}}{h^2} = \frac{\partial r^2}{\partial \eta}\frac{\partial \odot}{\partial \eta} - \frac{\partial r^2}{\partial \xi}\cdot\frac{\partial \odot}{\partial \xi} + \frac{4\odot}{h^2} + F \tag{1.4}$$

$$\frac{8\breve{\eta\eta}}{h^2} = \frac{\partial r^2}{\partial \xi} \cdot \frac{\partial \odot}{\partial \xi} - \frac{\partial r^2}{\partial \eta} \cdot \frac{\partial \odot}{\partial \eta} + \frac{4\odot}{h^2} - F \tag{1.5}$$

$$\frac{8\breve{\xi\eta}}{h^2} = -\frac{\partial r^2}{\partial \eta} \cdot \frac{\partial \odot}{\partial \xi} - \frac{\partial r^2}{\partial \xi} \frac{\partial \odot}{\partial \eta} - G \tag{1.6}$$

where $\odot = \breve{xx} + \breve{yy} = \breve{\xi\xi} + \breve{\eta\eta}$ (a plane harmonic function) and F and G are conjugate harmonic functions to be determined.

In the present case for the boundary $\xi = 0$, we have

$$\left.\begin{aligned} \left(\frac{\partial r^2}{\partial \xi}\right)_{\xi=0} &= 2c^2 \cos \eta, \left(\frac{\partial r^2}{\partial \eta}\right)_{\xi=0} = -2c^2 \sin \eta \\ \text{and} \qquad \left(\frac{1}{h^2}\right)_{\xi=0} &= \frac{c^2}{2\cos \eta} \end{aligned}\right\} \tag{1.7}$$

2. Solution for a uniform shear S at large distances from the hole. Suppose the plate is subject to a uniform shear S at large distances from the hole, parallel to the x and y axes

In this case we shall have

$$\left.\begin{aligned} (\breve{\xi\xi})_{\xi\to\infty} &= \left[2 S h^2 \frac{\partial x}{\partial \xi} \frac{\partial \eta}{\partial \xi}\right]_{\xi\to\infty} \\ (\breve{\eta\eta})_{\xi\to\infty} &= \left[-2 S h^2 \frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \xi}\right]_{\xi\to\infty} \\ (\breve{\xi\eta})_{\xi\to\infty} &= \left[S h^2 \left\{\left(\frac{\partial x}{\partial \xi}\right)^2 - \left(\frac{\partial y}{\partial \xi}\right)^2\right\}\right]_{\xi\to\infty} \end{aligned}\right\} \tag{2.1}$$

In the equation (1.4)—(1.6) we put $F = F_1 + F_2$ and $G = G_1 + G_2$ such that

$$F_1 + iG_1 = -8Si\left(\frac{\partial z}{\partial \zeta}\right)^2 \tag{2.2}$$

and
$$(F_2 h^2)_{\xi\to\infty} = 0 \quad \text{and} \quad (G_2 h^2)_{\xi\to\infty} = 0 \tag{2.3}$$

It is evident that if the contributions of all terms other than F_1 and G_1 on the right hand sides of (1.4) – (1.6) be zero, at large distances from the hole, then the stresses at infinity as given in (2.1) are obtained from the assumption (2.2).

On the boundary $\xi = 0$, we have

$$(F_1)_{\xi=0} = \frac{4Sc^2 \sin 3\eta}{\cos\eta}, \quad (G_1)_{\xi=0} = -\frac{4Sc^2 \cos 3\eta}{\cos\eta} \tag{2.4}$$

We assume
$$\odot = Ae^{-\xi} \sin \eta + Be^{-3\xi} \sin 3\eta \tag{2.5}$$
where A and B are real constants to be determined.

Then substituting the above value of $\odot$ into equations (1.4) and (1.6) and using the results (1.7) and (2.4), we obtain from the conditions

$$(\widetilde{\xi\xi})_{\xi=0} = 0 \quad \text{and} \quad (\widetilde{\xi\eta})_{\xi=0} = 0$$

the relations

$$\left.\begin{aligned}(F_2)_{\xi=0}\cos\eta &= -[4S+5B]c^2\sin 3\eta-(3B+2A)c^2\sin\eta\\ (G_2)_{\xi=0}\cos\eta &= [4S-3B]c^2\cos 3\eta-(3B+2A)c^2\cos\eta\end{aligned}\right\} \tag{2.6}$$

Obvious expressions for F_2 and G_2 will be given by

$$F_2+iG_2 = 2iQe^{-2\zeta} = 2iQe^{-2\xi}(\cos 2\eta - i\sin 2\eta) \tag{2.7}$$

where Q is a real constant to be determined.

Substituting the values of $(F_2)_{\xi=0}$ and $(G_2)_{\xi=0}$ from (2.7) into (2.6) and equating the coefficients of cosines and sines of like multiples of η on both sides of it, we get, a set of equations to determine the unknown constants. Thus

$$Q = 16Sc^2, \quad A = -2S, \quad B = -4S. \tag{2.8}$$

Hence, $\odot$, F and G are known and the stresses are completely determined.

On the boundary of the hole, we have

$$(\widetilde{\eta\eta})_{\xi=0} = [\odot]_{\xi=0} = -2S(\sin\eta+2\sin 3\eta)$$

At the node of the loop where $\eta = \pm\frac{1}{2}\pi$, this stress component has the value $\pm 2S$ while at the flat end where $\eta = 0$, it vanishes.

3. Solution for a uniform tension T parallel to the x-axis at large distances from the hole. Suppose that a uniform tension T is applied to the plate at infinity parallel to the x-axis Stress components at infinity, in this case, will be represented by

$$\left.\begin{aligned}(\widetilde{\xi\xi})_{\xi\to\infty} &= \frac{T}{2}\left[1+h^2\left\{\left(\frac{\partial x}{\partial\xi}\right)^2-\left(\frac{\partial y}{\partial\xi}\right)^2\right\}\right]_{\xi\to\infty}\\ (\widetilde{\eta\eta})_{\xi\to\infty} &= \frac{T}{2}\left[1-h^2\left\{\left(\frac{\partial x}{\partial\xi}\right)^2-\left(\frac{\partial y}{\partial\xi}\right)^2\right\}\right]_{\xi\to\infty}\\ (\widetilde{\xi\eta})_{\xi\to\infty} &= -T\left(h^2\frac{\partial x}{\partial\xi}\frac{\partial y}{\partial\xi}\right)_{\xi\to\infty}\end{aligned}\right\} \tag{3.1}$$

Let us now put $\odot = \odot_1+\odot_2$, $F = F_1+F_2$ and $G = G_1+G_2$, such that

$$\left.\begin{aligned}\odot_1 = T,\ F_1+iG_1 = 4T\left[\frac{\partial z}{\partial\zeta}\right]^2\\ [\odot_2]_{\xi\to\infty} = 0,\ [F_2h^2]_{\xi\to\infty} = 0,\ [G_2h^2]_{\xi\to\infty} = 0.\end{aligned}\right\} \tag{3.2}$$

Then the contributions of all terms excepting those of F_1, G_1 and $\odot_1$ being zero at infinity, the above assumptions will give the stresses, at large distances from the hole.

In the present case, we have

$$[F_1]_{\xi=0} = \frac{2Tc^2 \cos 3\eta}{\cos \eta}, \quad [G_1]_{\xi=0} = \frac{2Tc^2 \sin 3\eta}{\cos \eta}. \tag{3.4}$$

We assume

$$\odot_2 = Te^{-2\xi} \cos 2\eta + Be^{-3\xi} \cos 3\eta.$$

so that

$$\odot = \odot_1 + \odot_2 = T[1+e^{-2\xi} \cos 2\eta] + Be^{-3\xi} \cos 3\eta \tag{3.4}$$

where B is a real constant to be determined.

Substituting this value of $\odot$ into (1.4) and (1.6) and using the results (1.7) and (3.3) we obtain from the conditions $[\widetilde{\xi\xi}]_{\xi=0}$ and $[\widetilde{\xi\eta}]_{\xi=0} = 0$, the relations

$$\left.\begin{aligned} [F_2]_{\xi=0} \cos \eta &= -3Bc^2 \cos \eta - 4Tc^2(1+\cos 2\eta) - (5B+2T)c^2 \cos 3\eta \\ [G_2]_{\xi=0} \cos \eta &= 3Bc^2 \sin \eta + 2Tc^2 \sin 2\eta + (3B-2T)c^2 \sin 3\eta. \end{aligned}\right\} \tag{3.5}$$

If we assume

$$F_2 + iG_2 = \frac{2P}{1+e^{2\zeta}} + Qe^{\zeta} + Re^{-\zeta} - 2Me^{-2\zeta}, \tag{3.6}$$

where P, Q, R, M are real constants, we find

$$\left.\begin{aligned} [F_2]_{\xi=0} \cos \eta &= P \cos \eta + \tfrac{1}{2}(Q+R)(1+\cos 2\eta) - M(\cos 3\eta + \cos \eta) \\ [G_2]_{\xi=0} \cos \eta &= -P \sin \eta + \tfrac{1}{2}(Q-R) \sin 2\eta + M(\sin 3\eta + \sin \eta) \end{aligned}\right\} \tag{3.7}$$

Equating the coefficients of sines and cosines of like multiples of η in the expressions (3.5) and (3.7), we find

$$\left.\begin{aligned} B = -2T, \quad P = -2Tc^2, \quad Q = -2Tc^2 \\ R = -6Tc^2, \quad M = -8Tc^2. \end{aligned}\right\} \tag{3.8}$$

The unknown constants being determined all stress components are completely known from (1.4)—(1.6).

On the boundary of the hole, we have

$$[\widetilde{\eta\eta}]_{\xi=0} = [\odot]_{\xi=0} = T[1+\cos 2\eta - 2\cos 3\eta] \tag{3.9}$$

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Reference

Sen B. (1948), *Phil. Mag.*, Ser, 7, Vol **XXXIX**, 992.

ON THE JUMP OF A FUNCTION AND ITS FOURIER CO-EFFICIENTS

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(*Received— October 28, 1953.*)

1. Let $f(t)$ be integrable L in $(-\pi, \pi)$ and periodic with period 2π, and let

$$f(t) \sim \tfrac{1}{2}a_0 + \sum_1^{\infty}(a_n \cos nt + b_n \sin nt) = \tfrac{1}{2}a_0 + \sum_1^{\infty} A_n(t). \tag{1.1}$$

Then the conjugate series of (1.1) at $t = x$ is

$$\sum_1^{\infty}(b_n \cos nx - a_n \sin nx) = \sum_1^{\infty} B_n(x). \tag{1.2}$$

We write

$$\theta(t) = f(x+t) - f(x-t),\ \psi(t) = f(x+t) - f(x-t) - d(x), \text{ where } d(x) \text{ is finite,} \tag{1.3}$$

and

$$\Psi(t) = \int_0^t |\psi(u)|\,du.$$

The object of the present note is to prove

Theorem 1. *If*

$$\int_h^{\delta} \left| \frac{\psi(t)}{t} - \frac{\psi(t+h)}{t+h} \right| dt \to 0 \tag{1.4}$$

for some fixed δ, *when* $h \to +0$, *then the sequence* $\{nB_n(x)\}$ *is summable* $(C, 1)$ *to* $d(x)/\pi$.

It is known that the conclusion of Theorem 1 remains valid if (1.4) is replaced by one of the following :

(a) $\psi(t)$ is of bounded variation (Zygmund, 1935, p. 62, ex. 10) *

(b) $\psi_1(t) = \dfrac{1}{t}\displaystyle\int_0^t \psi(u)\,du$ is of bounded variation in $(0, \pi)$. **

2. Gergen (1930) has shown that condition (1.4) implies

$$\lim_{t\to 0} \frac{1}{t}\int_0^t |\psi(u)|\,du = 0, \text{***} \tag{2.1}$$

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* In this case the sequence $(nB_n(x))$ is summable (C, r) to $d(x)/\pi$, for r positive. This is analogous to Jordan's test for the convergence of Fourier Series and was given by Fejér (1913).

** This is the case $\alpha=1$ of a result due to H. C. Chow (1943), and is analogous to de la Vallée-Poussin's test for the convergence of Fourier Series. Theorem 1 of the present note corresponds to Lebesgue's test.

*** Condition (2.1) alone implies the summability $(C, 1+\delta)$, δ positive of the sequence $(nB_n(x))$ to $d(x)/\pi$.

so that in view of (1.4), we shall assume in what follows that condition (2.1) is also satisfied.

$\therefore$ We are to prove that

$$\frac{1}{n}\sum_{1}^{n} rB_r(x) \to d(x)/\pi \quad \text{as} \quad n\to\infty.$$

Writing $t_n(x) = \sum_{1}^{n} rB_r(x)$, we have

$$t_n(x)/n = \frac{1}{\pi}\int_0^{\pi} \{f(x+t)-f(x-t)\}\, g(n, t)\, dt,$$

where

$$g(n, t) = \frac{1}{n}\sum_{1}^{n} r \sin rt.$$

Denoting $\int_0^{\pi} g(n, t)\, dt$ by λ_n, we have

$$\lambda_n = \frac{1}{n}\sum_{1}^{n}\{1-(-1)^r\} = 1+o(1), \quad \text{as} \quad n\to\infty.$$

Therefore

$$t_n(x)/n - \lambda_n\, d(x)/\pi = \frac{1}{\pi}\int_0^{\pi}\{f(x+t)-f(x-t)-d(x)\}\, g(n, t)\, dt$$

$$= \frac{1}{\pi}\int_0^{\pi}\psi(t)\, g(n, t)\, dt.$$

Thus

$$t_n(x)/n - d(x)/\pi = \frac{1}{\pi}\int_0^{\pi}\psi(t)\, g(n, t)\, dt + o(1)$$

$$= J_1 + o(1), \quad \text{say.} \tag{2.2}$$

We have

$$g(n, t) = -\frac{1}{n}\frac{d}{dt}\{\tfrac{1}{2}+\cos t+\cos 2t+\ldots+\cos nt\}$$

$$= -\frac{1}{n}\frac{d}{dt}\left\{\frac{\sin(n+\frac{1}{2})t}{2\sin\frac{1}{2}t}\right\}$$

$$= \frac{1}{4n}\,\frac{\sin(n+\frac{1}{2})t\cos\frac{1}{2}t}{\sin^2\frac{1}{2}t} - \frac{n+\frac{1}{2}}{2n}\,\frac{\cos(n+\frac{1}{2})t}{\sin\frac{1}{2}t}.$$

Let $J = \int_0^{\pi}\psi(t) g(n, t)\, dt = \int_0^{h} + \int_h^{\pi} = J_1 + J_2$, say, where $h = \pi/n$. (2.3)

Now

$$g(n, t) = \frac{\sin(n+\frac{1}{2})t\cos\frac{1}{2}t - 2(n+\frac{1}{2})\cos(n+\frac{1}{2})t\sin\frac{1}{2}t}{4n\sin^2\frac{1}{2}t}.$$

By replacing the sines and cosines in the numerator of the above fraction by their respective power series expansions it is easy to see that, for $0 < t < \pi/n$, the numerator is $O(n^3t^3/\pi^3)$, and hence

$$g(n, t) = O(n^2t). \tag{2.4}$$

Using (2.1) and (2.4) we find

$$J_1 = \int_0^h \psi(t)g(n, t)\,dt = \int_0^h o(t)O(n^2t)\,dt = o(1). \tag{2.5}$$

Also
$$J_2 = \int_h^\pi \psi(t)\left\{\frac{\sin(n+\frac{1}{2})t\cos\frac{1}{2}t}{4n\sin^2\frac{1}{2}t} - \frac{n+\frac{1}{2}}{2n}\,\frac{\cos(n+\frac{1}{2})t}{\sin\frac{1}{2}t}\right\}dt$$

$$= P - Q, \text{ say}. \tag{2.6}$$

Now
$$|P| = \left|\int_h^\pi \psi(t)\,\frac{\sin(n+\frac{1}{2})t\cos\frac{1}{2}t}{4n\sin^2\frac{1}{2}t}\,dt\right|$$

$$\leq \frac{1}{4n}\int_h^\pi \frac{|\psi(t)|}{\sin^2\frac{1}{2}t}\,dt$$

$$< \frac{\pi^2}{4n}\int_h^\pi \frac{|\psi(t)|}{t^2}\,dt$$

$$\leq \frac{\pi^2}{4n}\,\frac{\Psi(\pi)}{\pi^2} + \frac{\pi^2}{2n}\int_h^\pi \frac{\Psi(t)}{t^3}\,dt$$

$$= o(1) + O(1/n)\int_h^\pi o(1/t^2)\,dt = o(1). \tag{2.7}$$

Hence
$$P = o(1).$$

Again
$$Q = \frac{n+\frac{1}{2}}{n}\int_h^\pi \psi(t)\,\frac{\cos(n+\frac{1}{2})t}{2\sin\frac{1}{2}t}\,dt$$

$$= \frac{n+\frac{1}{2}}{n}R, \text{ say}, \tag{2.8}$$

where
$$R = \int_{\pi/n}^\pi \psi(t)\left\{\frac{\cos nt\cos\frac{1}{2}t}{2\sin\frac{1}{2}t} - \frac{\sin nt\sin\frac{1}{2}t}{2\sin\frac{1}{2}t}\right\}dt$$

$$= \tfrac{1}{2}\int_{\pi/n}^\pi \psi(t)\cot\tfrac{1}{2}t\cos nt\,dt - \tfrac{1}{2}\int_{\pi/n}^\pi \psi(t)\sin nt\,dt$$

$$= \tfrac{1}{2}\int_{\pi/n}^\pi \psi(t)\{\cot\tfrac{1}{2}t - 2/t\}\cos nt\,dt + \int_{\pi/n}^\pi \frac{\psi(t)}{t}\cos nt\,dt - \tfrac{1}{2}\int_{\pi/n}^\pi \psi(t)\sin nt\,dt$$

$$= \int_{\pi/n}^{\delta} \frac{\psi(t)}{t} \cos nt\,dt + \int_{\delta}^{\pi} \frac{\psi(t)}{t} \cos nt\,dt + \tfrac{1}{2}\int_{\pi/n}^{\pi} \psi(t)\,\{\cot \tfrac{1}{2}t - 2/t\} \cos nt\,dt$$

$$- \tfrac{1}{2}\int_{\pi/n}^{\pi} \psi(t) \sin nt\,dt$$

where δ is a constant such that $\pi/n < \delta < \pi$. The last three integrals above tend to zero as $n \to \infty$. Hence we get

$$R = \int_{\pi/n}^{\delta} \frac{\psi(t)}{t} \cos nt\,dt + o(1)$$

$$= \int_{\pi/n}^{2\pi/n} \frac{\psi(t)}{t} \cos nt\,dt + \int_{2\pi/n}^{\delta+\pi/n} \frac{\psi(t)}{t} \cos nt\,dt - \int_{\delta}^{\delta+\pi/n} \frac{\psi(t)}{t} \cos nt\,dt + o(1)$$

$$= \int_{\pi/n}^{2\pi/n} \frac{\psi(t)}{t} \cos nt\,dt - \int_{\pi/n}^{\delta} \frac{\psi(t+\pi/n)}{t+\pi/n} \cos nt\,dt - \int_{\delta}^{\delta+\pi/n} \frac{\psi(t)}{t} \cos nt\,dt + o(1)$$

Therefore

$$2R = \int_{\pi/n}^{2\pi/n} \frac{\psi(t)}{t} \cos nt\,dt - \int_{\delta}^{\delta+\pi/n} \frac{\psi(t)}{t} \cos nt\,dt + \int_{\pi/n}^{\delta} \left\{\frac{\psi(t)}{t} - \frac{\psi(t+\pi/n)}{t+\pi/n)}\right\} \cos nt\,dt + o(1)$$

$$= R_1 - R_2 + R_3 + o(1), \text{ say.}$$

Now obviously $$R_2 = o(1). \tag{2.9}$$

Also, by (2.1), we have

$$R_1 = -\int_0^{\pi/n} \frac{\psi(t+\pi/n)}{t+\pi/n} \cos nt\,dt \leq \int_0^{\pi/n} \frac{|\psi(t+\pi/n)|}{t+\pi/n} dt$$

$$= \int_{\pi/n}^{2\pi/n} \frac{|\psi(t)|}{t} dt = \frac{n}{\pi} \int_{\pi/n}^{\tau} |\psi(t)|\,dt, \text{ where } \pi/n < \tau < 2\pi/n$$

$$\leq \frac{n}{\pi} \int_0^{2\pi/n} |\psi(t)|\,dt = o(1)$$

Thus $$R_1 = o(1) \tag{2.10}$$

Again, by (1.4), we have

$$|R_3| \leq \int_{\pi/n}^{\delta} \left|\frac{\psi(t)}{t} - \frac{\psi(t+\pi/n)}{t+\pi/n}\right| dt = o(1) \tag{2.11}$$

Thus by (2.9), (2.10) and (2.11)

$$R = o(1) \tag{2.12}$$

whence by (2.12) $$Q = \{(n+\tfrac{1}{2})/n\}\ R = o(1). \tag{2.13}$$

By (2.3), (2.5), (2.6), (2.7) and (2.13)

$$J = o(1). \tag{2.14}$$

Therefore by (2.2) and (2.14)

$$t_n(x)/n - d(x)/\pi = o(1) \quad \text{as} \quad n \to \infty,$$

and this completes the proof of the theorem.

3. Combining the above theorem with the case $K = 0$ of

Theorem A (Hardy and Littlewood, 1931) *If Σu_n is summable (A), then a necessary and sufficient condition that it should be summable (C, K), $K > -1$, is that the sequence $\{nu_n\}$ is summable $(C, K+1)$ to the value 0,*

we get

Theorem 2. *If*

$$(a) \qquad g(x) = \frac{1}{2\pi} \int_0^\pi \theta(t) \cot \tfrac{1}{2}t \, dt \tag{3.1}$$

exists as a Cauchy integral down to zero and

$$(b) \qquad \int_h^\delta \left| \frac{\theta(t+h)}{t+h} - \frac{\theta(t)}{t} \right| dt \to 0 \tag{3.2}$$

as $h \to +0$, for some fixed δ, then the allied series is convergent.

Proof of Theorem 2. The existence of the integral (3.1) as a Cauchy integral down to zero implies the summability (A) of the allied series (1.2). By using Theorem 1 we find that (3.2) implies summability $(C, 1)$ of the sequence $\{nB_n(x)\}$ to the value zero. Now the convergence of the allied series is a consequence of the case $K = 0$ of Theorem A.

It is relevant to note in this connection that the following more general theorem is also true and is the analogue for the conjugate series of the refinement by Gergen of Lebesgue's test for the convergence of Fourier Series.

Theorem 3. *If*

$$(a) \qquad g(x) = \frac{1}{2\pi} \int_0^\pi \theta(t) \cot \tfrac{1}{2}t \, dt$$

exists as a Cauchy integral down to zero and

$$(b) \qquad \int_h^\delta (|\theta(t+h) - \theta(t)|/t) \, dt \to 0$$

as $h \to +0$, for some fixed δ, then the allied series is convergent.

Proof of Theorem 3. Condition (a) of Theorem 3 implies that

$$H(t) = \int_0^t \frac{\theta(u)}{u} du \text{ exists as a Cauchy integral down to zero.} \tag{3.3}$$

Thus it follows that

$$\Theta(t) = \int_0^t \theta(u)\, du = \int_0^t uH'(u)\, du = tH(t) - \int_0^t H(u)\, du = o(t),$$

(Hardy and Rogosinski, 1950, p. 49). (3.4)

If (3.4) holds good, then it can be proved by standard arguments that (Hardy and Rogosinski, 1950, p. 49)

$$\int_0^{\pi/n} \theta(u) \frac{1-\cos nu}{u} du = o(1). \tag{3.5}$$

The same argument will prove that, if 'a' is any positive constant, then

$$\int_{\tau_1}^{\tau_2} \theta(u) \frac{1-\cos nu}{u} du = o(1), \quad \text{where } 0 \leq \tau_1 < \tau_2 \leq a/n. \tag{3.6}$$

In view of (3.3), condition (3.6) may be replaced by

$$\int_{\tau_1}^{\tau_2} \theta(u) \frac{\cos nu}{u}\, du = o(1) \tag{3.7}$$

Now

$$S_n = \sum_{m=1}^{n} (b_m \cos mx - a_m \sin mx)$$

$$= \frac{1}{\pi} \int_0^{\pi} \{f(x+t) - f(x-t)\} \frac{\sin \frac{1}{2}(n+1)t \sin \frac{1}{2} nt}{\sin \frac{1}{2} t}\, dt$$

$$= \frac{1}{2\pi} \int_0^{\pi} \theta(t) \cot \tfrac{1}{2}t\, (1-\cos nt)\, dt + \frac{1}{2\pi} \int_0^{\pi} \theta(t) \sin nt\, dt.$$

Obviously, the second integral on the right $= o(1)$.

Hence

$$S_n - \frac{1}{2\pi} \int_h^{\pi} \theta(t) \cot \tfrac{1}{2}t\, dt$$

$$= \frac{1}{2\pi} \int_0^{h} \theta(t) \cot \tfrac{1}{2}t\, (1-\cos nt)\, dt - \frac{1}{2\pi} \int_h^{\pi} \theta(t) \cot \tfrac{1}{2}t \cos nt\, dt + o(1)$$

$$= P - Q + o(1), \text{ say.}$$

Now

$$P = \frac{1}{2\pi} \int_0^{h} \theta(t) \cot \tfrac{1}{2}t (1-\cos nt)\, dt$$

$$= \frac{1}{\pi} \Theta\left(\frac{\pi}{n}\right) \cot \frac{\pi}{2\pi} - \int_0^{h} \Theta(t) \frac{d}{dt}\left(\frac{1-\cos nt}{\tan \frac{1}{2}t}\right) dt$$

and the integrated term $= o(1)$. Also $\Theta(t) = o(t)$ by (3.4),

and
$$\left|\frac{d}{dt}\left(\frac{1-\cos nt}{\tan \frac{1}{2}t}\right)\right| < A\left(\frac{n}{t}+n^2\right).$$

Hence
$$P = o\left(n\int_0^{\pi/n} dt\right)+o\left(n^2\int_0^{\pi/n} t\,dt\right) = o(1).$$

Again
$$\pi Q = \int_h^{\delta} \frac{\theta(t)}{t}\cos nt\,dt+\tfrac{1}{2}\int_h^{\delta} \theta(t)\left\{\cot\tfrac{1}{2}t-\frac{2}{t}\right\}\cos nt\,dt+\tfrac{1}{2}\int_{\delta}^{\pi} \theta(t)\cot\tfrac{1}{2}t\cos nt\,dt;$$

where δ is a constant such that $h<\delta<\pi$. The last two integrals above tend to zero when $n\to\infty$.

Hence it remains to consider the integral
$$\int_h^{\delta} \frac{\theta(t)}{t}\cos nt\,dt$$

Using (3 4), (3.7) and condition (b) of Theorem 3 given above, it can be proved by arguments precisely similar to those employed in the proof of Gergen's refinement of Lebesgue's Test for convergence of Fourier series (Hardy and Rogosinski, 1950, p. 43) that
$$\int_h^{\delta} \frac{\theta(t)}{t}\cos nt\,dt = o(1).$$

This would complete the proof of Theorem 3.

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References

Chow, H. C. (1942), *Jour. Lond. Math. Soc.*, **17**, 179.
Fejér, L. (1913), *Jour. f. reine u. angew. Math.*, **142**, 165
Gergen, (1930), *Quart. Jour. Math.* (Oxford), 252.
Hardy, G. H. and Littlewood, J. (1931), *Jour. Lond. Math Soc*, **6**, 283.
Hardy, G. H. and Rogosinski, W. W. (1950)' Fourier Series, 2nd Ed Cambridge
Zygmund, A. (1935), Trigonometrical Series, Warsaw

VELOCITY AND TEMPERATURE DISTRIBUTIONS IN A FORCED JET OF A COMPRESSIBLE FLUID

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(*Received—March 26, 1954*)

Introduction. The boundary layer equations of a compressible fluid have often been used by many authors for studying steady laminar flow in a plane jet. Illingworth (1949), Toose (1952) and Pai (1949) have studied the problem when the Prandtl number is unity and the pressure-gradient is negligible. In all these cases the flux of momentum across any transverse section of the jet is constant and the temperature distribution is at once given by Crocco's relation.

In the present paper though the Prandtl number has been taken as unity and the pressure-gradient negligible, it has been assumed that the jet is maintained by an extraneous force. The form of this force is suggested by Karman's Integral equation. Evidently in this case the flux of momemtum will not remain constant. Two methods have been suggested to completely specify the problem, either the extraneous force can be taken as constant along the axis of the jet, vanishing at its edge or the rate of the flux of momentum across any section may be taken as constant.

Method used is the usual transformation of the boundary layer equation with the help of Mises' transformation formulae and its solution with a suitable extraneous force. Here no such relation as one due to Crocco exists but the energy equation has been solved directly when the velocity distribution is known.

2. Basic Equations. Taking the x-axis along the axis of the jet and origin outside it, the boundary layer equation in the x-direction when, the pressure-gradient is neglected, is

$$\varrho u\frac{\partial u}{\partial x}+\varrho v\frac{\partial u}{\partial y}=\varrho X+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right), \tag{1}$$

where X is the extraneous force in the x-direction per unit mass.

The equation of continuity is

$$\frac{\partial}{\partial x}(\varrho u)+\frac{\partial}{\partial y}(\varrho v)=0. \tag{2}$$

When the Prandtl number is unity, the equation of energy is

$$\varrho u\frac{\partial i}{\partial x}+\varrho v\frac{\partial i}{\partial y}=\frac{\partial}{\partial y}\left(\mu\frac{\partial i}{\partial y}\right)+\mu\left(\frac{\partial u}{\partial y}\right)^2 \tag{3}$$

where i is the heat content.

When the pressure is constant, the equation of state for a perfect gas is

$$i\varrho = \text{const.} = i_1\varrho_1 \tag{4}$$

suffix 1 denoting the value at the edge of the jet

We further assume that

$$\mu/\mu_1 = i/i_1$$

The boundary conditions are

on the axis of the jet, $\quad y = 0, \quad v = 0, \; \dfrac{\partial u}{\partial y} = 0, \quad (6)$

at the edge. $\quad y = \infty, \quad u = 0.$

3 **Extraneous Force.** Karman's integral equation for the boundary-layer in the present case will reduce to

$$\frac{\partial}{\partial x}\int_0^\infty \varrho u^2\,dy = \int_0^\infty \varrho X dy. \tag{8}$$

As the integral on the left-hand side will vary as some power of x, we can take

$$X = cu^2/x \tag{9}$$

when c is some constant.

4. **Simplifications of the equations.** With the above value of X, the equation (1) becomes

$$\varrho u\frac{\partial u}{\partial x} + \varrho v\frac{\partial u}{\partial y} = c\varrho\frac{u^2}{x} + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \tag{10}$$

The equation (2) is satisfied by

$$\varrho u = \partial\psi/\partial y, \quad \varrho v = -\partial\psi/\partial x. \tag{11}$$

We now change the independent variables (x, y) to (x, ψ) by means of the transformation formulae

$$\left.\begin{aligned} (\partial/\partial y)_x &= \varrho u(\partial/\partial\psi)_x \\ (\partial/\partial x)_y &= \partial/\partial x)_\psi - \varrho v(\partial/\partial\psi)_x. \end{aligned}\right\} \tag{12}$$

The equation (10) then becomes

$$\frac{\partial u}{\partial x} = \frac{cu}{x} + \frac{\partial}{\partial\psi}\left(\mu\varrho u\frac{\partial u}{\partial\psi}\right). \tag{13}$$

But from (4) and (5) we have

$$\mu\varrho = \mu_1\varrho_1$$

hence the above equation becomes

$$\frac{\partial u}{\partial x} = \frac{cu}{x} + \mu_1\varrho_1\frac{\partial}{\partial\psi}\left(u\frac{\partial u}{\partial\psi}\right). \tag{13}$$

Take $$\eta = (\mu_1\varrho_1)^{-\frac{1}{2}}x^n\psi, \quad u = x^m f(\eta)$$

so that (13) gives

$$mx^{m-1}f(\eta)+nx^{m-1}\eta f'(\eta) = cx^{m-1}f(\eta)+x^{2(m+n)}\frac{d}{d\eta}\{f(\eta)f'(\eta)\}, \tag{14}$$

where dash means differentiation with respect to η.

In order that (14) may be a differential equation involving η only, we must have

$$m-1 = 2(m+n) \text{ or } m+2n = -1. \tag{15}$$

Hence (14) becomes

$$-(1+2n)f+n\eta f' = cf+\frac{d}{d\eta}(ff'),$$

or $$-(1+c+2n)f+n\eta f' = \frac{d}{d\eta}(ff'). \tag{16}$$

Before we go to solve equation (16) we see what conditions the function $f(\eta)$ satisfies.

5. Solution of the equation. From definition we have

$$d\psi = (\partial\psi/\partial x)_\eta dx+(\partial\psi/\partial\eta)_x d\eta = (\partial\psi/\partial y)_x dy+(\partial\psi/\partial x)_y dx$$

where $$\psi = (\mu_1\varrho_1)^{\frac{1}{2}}x^l\eta$$

taking $$l = -n. \tag{17}$$

Thus $$l(\mu_1\varrho_1)^{\frac{1}{2}}x^{l-1}\eta dx+(\mu_1\varrho_1)^{\frac{1}{2}}x^l d\eta = \varrho u dy-\varrho v dx$$

giving $$\varrho v = -l(\mu_1\varrho_1)^{\frac{1}{2}}x^{l-1}\eta$$

and $$\varrho u dy = (\mu_1\varrho_1)^{\frac{1}{2}}x^l d\eta. \tag{19}$$

From (18) we see that $v = 0$ when $\eta = 0$ hence $\eta = 0$ corresponds to $y = 0$.

Further $$\partial u/\partial y = \varrho u\partial u/\partial\psi = \varrho u x^m f'(\eta).\ (\mu_1\varrho_1)^{-\frac{1}{2}}x^n,$$

hence the condition $$\frac{\partial u}{\partial y} = 0 \text{ when } y = 0$$

gives $$f'(\eta) = 0 \text{ when } \eta = 0.$$

Thus the function $f(\eta)$ satisfies the conditions,

on the axis of the jet $\eta = 0$, $\qquad f'(\eta) = 0. \qquad (20)$

at the edge $\qquad f(\eta) = 0.$

We are now in a position to solve the equation (15).

Put $$n = -(1+c+2n) \quad i.e.\ 1+c+3n = 0, \tag{22}$$

so that equation (16) becomes

$$n(f+\eta f') = \frac{d}{d\eta}(ff')$$

or from (17),
$$-l(f+\eta f') = \frac{d}{d\eta}(ff')$$

which gives on integration
$$-l\eta f = ff'$$

since
$$f' = 0 \text{ when } \eta = 0, \quad \text{from (20).}$$

Thus further integration gives
$$f = c-\tfrac{1}{2}l\eta^2 \tag{23}$$

Let $\eta = \eta_0$, give the edge of the jet so that from (21), we have
$$f = \tfrac{1}{2}l\eta_0^2\{1-(\eta/\eta_0)^2\} \tag{24}$$

Thus
$$u = \tfrac{1}{2}lx^m\eta_0^2\{1-(\eta/\eta_0)^2\}$$

or
$$u = U\,\{1-(\eta/\eta_0)^2\} \tag{25}$$

when
$$U = \tfrac{1}{2}lx^m\eta_0^2 \tag{26}$$

Evidently U is the velocity on the axis of the jet.

6. Solution of the energy equation The equation (3) with the help of the transformation (12) becomes
$$\frac{\partial i}{\partial x} = \mu_1\varrho_1\left[\frac{\partial}{\partial\psi}\left(u\frac{\partial i}{\partial\psi}\right)+u\left(\frac{\partial u}{\partial\psi}\right)^2\right]. \tag{27}$$

To solve this equation we now put
$$i = i_1+x^tF(\eta), \tag{28}$$

where η is the same as before, i.e. $\eta = (\mu_1\varrho_1)^{-\frac{1}{2}}x^{-l}\psi$.

Remembering that
$$u = \tfrac{1}{2}lx^m(\eta_0^2-\eta^2)$$

the equation (27) becomes
$$tx^{t-1}F-lx^{t-1}\eta F' = \tfrac{1}{2}lx^{m+t-2l}(\eta_0^2-\eta^2)F''$$
$$-lx^{m+t-2l}\eta F'+\tfrac{1}{2}l^3x^{3m-2l}(\eta_0^2-\eta^2)\eta^2,$$

so that
$$t-1 = m+t-2l = 3m-2l$$

giving
$$m-2l = -1, \qquad \text{same as (15)}$$

and
$$t = 2m \tag{29}$$

and the equation for $F(\eta)$ is
$$4mF = l(\eta_0^2-\eta^2)(F''+l^2\eta^2). \tag{30}$$

The solution of the equation (30) is found to be
$$F = \frac{l^3}{4(2m+l)(m+3l)}(\eta_0^2-\eta^2)\{l\eta_0^2+(2m+l)\eta^2\} \tag{31}$$

so that
$$i = i_1+\frac{l^3x^{2m}}{4(2m+l)(m+3l)}(\eta_0^2-\eta^2)\{l\eta_0^2+(2m+l)\eta^2\}. \tag{32}$$

7. Transformation formula We now complete the transformation formula (19)

$$dy = (\mu_1\varrho_1)^{\frac{1}{2}}\frac{x^l}{\varrho u}d\eta = \frac{(\mu_1\varrho_1)^{\frac{1}{2}}}{i_1\varrho_1}\frac{ix^l}{u}d\eta$$

$$= \frac{1}{i_1}\left(\frac{\mu_1}{\varrho_1}\right)^{\frac{1}{2}}x^l\left[\frac{2i_1x^{-m}}{l(\eta_0^2-\eta^2)} + \frac{l^3x^m}{2(2m+l)(m+3l)}\{l\eta_0^2+(2m+l)\eta^2\}\right]d\eta$$

$$\therefore\ y = \left(\frac{\mu_1}{\varrho_1}\right)^{\frac{1}{2}}x^{l-m}\left[\frac{2}{l\eta_0}\tanh^{-1}\left(\frac{\eta}{\eta_0}\right) + \frac{l^2x^{2m}\eta}{6i_1(2m+l)(m+3l)}\{3l\eta_0^2+(2m+l)\eta^2\}\right] \qquad (33)$$

as $\qquad y = 0$ when $\eta = 0$

Thus $\qquad \eta = \eta_0$ corresponds to $y = \infty$.

8. Determination of the constants Two methods have been suggested In the first method, we make the extraneous force constant along the axis of the jet.

Now $$X = cu^2/x = \tfrac{1}{4}cl^2x^{2m-1}(\eta_0^2-\eta^2)^2 \qquad (34)$$

so that on the axis of the jet where $\eta = 0$, we have

$$X_0 = \tfrac{1}{4}cl^2x^{2m-1}\eta_0^4 = K, \text{ say}, \qquad (35)$$

so that $$m = \tfrac{1}{2} \text{ and } \eta_0^4 = 4K/(cl^4) \qquad (36)$$

With this value of m, we find from (15), (17) and (22)

$$l = \frac{3}{4},\ c = \frac{5}{4}, \qquad (37)$$

and $$\eta_0^2 = \frac{16}{3}\left(\frac{k}{5}\right)^{\frac{1}{2}}. \qquad (38)$$

These give
$$\left.\begin{aligned} u &= 3x^{\frac{1}{4}}(\eta_0^2-\eta^2)/8,\quad \eta = (\mu_1\varrho_1)^{-\frac{1}{2}}x^{-\frac{3}{4}}\psi.\\ X &= K\{1-(\eta/\eta_0)^2\},\\ i &= i_1 + \frac{27x}{4928}(\eta_0^2-\eta^2)(3\eta_0^2+7\eta^2),\\ y &= \left(\frac{\mu_1}{\varrho_1}\right)^{\frac{1}{2}}x^{\frac{1}{4}}\left[\frac{8}{3\eta_0}\tanh^{-1}\left(\frac{\eta}{\eta_0}\right) + \frac{3x}{616i_1}\eta(9\eta_0^2+7\eta^2)\right) \end{aligned}\right\} \qquad (39)$$

Also the flux of momentum across any transverse section is

$$2\int_0^{\eta_0}\varrho u^2dy = \tfrac{8}{3}l(\mu_1\varrho_1)^{\frac{1}{2}}x^{l+m}\eta_0^3 \qquad (40)$$

and this becomes $$\tfrac{1}{2}(\mu_1\varrho_1)^{\frac{1}{2}}\eta_0^3x^{5/4}. \qquad (41)$$

In the second method, we assume that the flux of momentum is Kx where K is a constant,

so that from (40) we have $$l+m = 1,\ \tfrac{8}{3}l(\mu_1\varrho_1)^{\frac{1}{2}}\eta_0^3 = K \qquad (42)$$

then we get $$l = \tfrac{2}{3},\ m = \tfrac{1}{3},\ c = 1 \qquad (43)$$

and $$\eta_0^3 = \frac{9K}{4(\mu_1\varrho_2)^{\frac{1}{2}}} \qquad (44)$$

These give

$$\left.\begin{aligned} u &= \tfrac{1}{3}x^{\frac{1}{3}}(\eta_0^2-\eta^2), \quad \eta = (\mu_1\varrho_1)^{-\frac{1}{2}}x^{-\frac{2}{3}}\psi, \\ X &= \tfrac{1}{9}\eta_0^4 x^{-\frac{1}{3}}\left\{1-\left(\frac{\eta}{\eta_0}\right)^2\right\}^2, \\ i &= i_1+\tfrac{1}{63}x^{\frac{2}{3}}(\eta_0^2-\eta^2)(\eta_0^2+2\eta^2), \\ y &= \left(\frac{\mu_1}{\varrho_1}\right)^{\frac{1}{2}}x^{\frac{1}{3}}\left[\frac{3}{\eta_0}\tanh^{-1}\left(\frac{\eta}{\eta_0}\right)+\frac{1}{63i_1}x^{\frac{2}{3}}(\eta_0^2+2\eta^2).\right] \end{aligned}\right\} \tag{45}$$

9. Results. Flux of mass across any transverse section is

$$2\int_0^{\eta_0}\varrho u dy = 2(\mu_1\varrho_1)^{\frac{1}{2}}x^l\eta_0 \tag{46}$$

More and more ambient fluid is drawn into the jet as it progresses.

In the first case, the extraneous force is constant along the axis of the jet and vanishes at its edge. This is a more natural case for such a force if any to exist. Velocity on the axis is of the order $x^{\frac{1}{2}}$.

In the second case, the extraneous force is of the order $x^{-\frac{1}{3}}$ and therefore vanishes for large values of x but is large for small values of x, while the velocity on the axis is of the order $x^{\frac{1}{3}}$.

In both the case the flow is helped by the extraneous force so that the flux of momentum increases as we go down the jet.

10. Incompressible liquid. In this case $\varrho = \varrho_1$, therefore writing ϱ for ϱ_1 and μ for μ_1, we get from (19)

$$dy = \left(\frac{\mu}{\varrho}\right)^{\frac{1}{2}}\frac{x^l}{u}d\eta = \frac{2}{l}\left(\frac{\mu}{\varrho}\right)^{\frac{1}{2}}\frac{x^{l-m}}{\eta_0^2-\eta^2}d\eta$$

$$\therefore\ y = \frac{2}{l\eta_0}\left(\frac{\mu}{\varrho}\right)^{\frac{1}{2}}x^{l-m}\tanh^{-1}\left(\frac{\eta}{\eta_0}\right) \quad \therefore\ \frac{\eta}{\eta_0} = \tanh\zeta \text{ where } \zeta = \frac{l\eta_0}{2}\left(\frac{\varrho}{\mu}\right)^{\frac{1}{2}}\frac{y}{x^{l-m}}$$

so that

$$u = \tfrac{1}{2}lx^m\eta_0^2\,\mathrm{sech}^2\,\zeta.$$

In the first case

$$\zeta = \frac{3}{2}\left(\frac{K\varrho^2}{45\mu^2}\right)^{\frac{1}{4}}\frac{y}{x^{\frac{1}{4}}}, \quad u = 2\left(\frac{Kx}{5}\right)^{\frac{1}{2}}\mathrm{sech}^2\,\zeta.$$

In the second case

$$\zeta = \tfrac{1}{3}\left(\frac{9K\varrho}{4\mu^2}\right)^{\frac{1}{3}}\frac{y}{x^{\frac{1}{3}}}, \quad u = \tfrac{1}{3}\left(\frac{81K^2x}{16\mu\varrho}\right)^{\frac{1}{3}}\mathrm{sech}^2\,\zeta.$$

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References

Illingworth, C. R. (1949). *Proc Roy. Soc.* (A) **199**, 533.
Pai, S. I. (1949). *J. Aero. Sci.* **16**, 463.
Toose D. G. (1952), *Quart. Mech & Appl. Math.* **5**, 155.

ZUR ELEKTROMAGNETIK MATERIELLER KÖRPER AUF GRUNDLAGE DER MASSFORMEL DES VIERERRAUMES

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(*Communicated by the Secretary—Received March 24, 1954*)

Inhaltsübersicht. Es werden unten die Grundlagen einer neuen Theorie des Elektromagnetismus materieller Korper vorgelegt, die von der diesbezüglichen Theorie von Minkowski prinzipiell abweicht. Ihre Hauptzüge sind die folgenden: (i) der sogenannte Brechungsindex der Materie gilt als Funktion des Gravitationsfeldes und (ii) die primären materiellen Konstanten für homogene isotrope Körper sind der Brechungsindex und ein gewisser Lorentzinvariante Skalar, woraus sich die Dielektrizitätskonstante und Permeabilität zusammensetzen. Das wertvolle gegenuber der Minkowski'schen Theorie ist u.a. die konkrete Formel (38) des Textes für die Kraftdichte.

1. Prinzipielle Erwägungen. Eine grundlegende Behauptnug der Gravitationslehre Einsteins (1922) ist die, dass sich die Lichtgeschwindigkeit zu einer beliebigen Raumzeitstelle und in einer beliebigen Richtung aus der Massformel

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

durch das einfache Nullsetzen von ds erhalten lasse. Umgekehrt muss also folgendes zutreffen. Falls sich die Lichtgeschwindigkeit zu einer beliebigen Raumzeitstelle und in einer beliebigen Richtung irgendwie etwa mit

$$0 = h_{\mu\nu} dx^\mu dx^\nu$$

angeben lässt, dann wird auch die Massformel zu dieser Stelle mit

$$ds^2 = f . h_{\mu\nu} dx^\mu dx^\nu$$

anzugeben sein bis auf einen unbekannten Faktor $f(x^4, x^1, x^2, x^3)$.

Es liege ein endliches Stück ruhender, homogener, isotroper Materie vor, der erfahrungsgemäss etwa der Brechungsindex n zuzuschreiben ist, in der, nämlich, die Lichtgeschwindigkeit zu jeder Stelle und in jeder Richung als c/n anzusetzen ist. Nach dem Gesagten wird dann die in diesem Stück herrschende Massformel ohne weiteres mit

$$ds^2 = f\left\{ \frac{c^2 dt^2}{n} - n\Sigma dx^2 \right\} \tag{1}$$

anzugeben sein bis auf den unbekannten Faktor f. Dem materiefreien Raum, welcher das betrachtete materielle Stück umgibt, muss man ebenso eine Massformel von der Gestalt

$$ds^2 = h\{c^2dt^2 - \Sigma dx^2\} \tag{2}$$

zuschreiben, weil dort die Lichtgeschwindigkeit zu jeder Stelle und in jeder Richtung c ist. Hier ist h ebenfalls eine bis auf weiteres nicht bekannte Funktion. Einstweilen sehen wir von jeglichen näheren Bestimmungen das Gravitationsgesetzes ab.

2. Betrachtungen betreffend den Mitführungskoeffizient von Fresnel. Befindet sich das Korperstück etwa mit der gleichformigen Geschwindigkit u nach der positiven x-Richtung bewegt, so fordern wir im Sinne der speziellen Relativitätstheorie, dass dann die vom Beobachter wahrgenommenen Verhältnisse der Lichtfortpflanzung aus denjenigen Massformeln zu entnehmen sein, welche aus (1) bzw. (2) durch die Lorentztransformtion hervorgehen. Nach ausgeführter Lorentztransformation gemäss

$$\left.\begin{aligned} x &= \beta(x'-ut'), \; y = y', \; z = z' \\ t &= \beta(t'-ux'/c^2)\,; \; \beta = (1-u^2/c^2)^{-\frac{1}{2}} \end{aligned}\right\} \tag{3}$$

geht (1) in

$$\begin{aligned} ds^2 &= f(x', y, z, t')n\left\{\left(\frac{c^2}{n^2}-c^2\right)\beta^2\left(dt'-\frac{udx'}{c^2}\right)^2 + c^2dt'^2 - dx'^2 - dy^2 - dz^2\right\} \\ &= f(x', y, z, t')n\left\{\frac{c^2}{n^2}dt'^2 + 2\left(1-\frac{1}{n^2}\right)udt'dx' - dx'^2 - dy^2 - dz^2\right\} \end{aligned} \tag{4}$$

über, falls dabei höhere Potenzen von u/c als die erste vernachlässigt werden Für den materiefreien Raumteil, der den Körper umgibt, geht die Massformel (2) in

$$ds^2 = h(x', y, z, t')\{c^2dt'^2 - dx'^2 - dy^2 - dz^2\} \tag{5}$$

über. Hieraus ergibt sich für den Beobachter die unveränderte Lichtgeschwindigkeit c worauf eben die Lorentztransformation auch angepasst ist. Aus (4) hat man für die Bestimmung der Lichtgeschwindigkeit im Inneren des bewegten Körperstückes die Gleichung

$$\left.\begin{aligned} 0 &= \frac{c^2}{n^2} + 2\left(1-\frac{1}{n^2}\right)u\frac{dx'}{dt'} - \frac{dx'^2+dy^2+dz^2}{dt'^2}, \\ &= \frac{c^2}{n^2} + 2\left(1-\frac{1}{n^2}\right)uV\cos\theta - V^2, \end{aligned}\right\} \tag{6}$$

wobei V die gesuchte Lichtgeschwindigkeit bezeichnet und θ den zwischen der Fortpflanzungsrichtung des Lichtes und der Bewegungsrichtung des Körpers einge-

schlossenen Winkel. Durch Lösung der quadratischen Gleichung (6), dabei aber unter Vernachlässigung höherer Potenzen von u/c als der ersten, erhalt man

$$V = \frac{c}{n} + \left(1 - \frac{1}{n^2}\right) u \cos\theta \tag{7}$$

und somit den richtigen Ausdruck des Fresnel'scher Mitführungskoeffizientes.

Die obenstehenden Überlegungen legen die folgende Ansicht ausserst nahe, welche wir hier auch vertreten wollen: Die Ablenkung des Lichts bei seinem Eintritt in die Materie ist im Grunde dieselbe Erscheinung wie die Ablenkung des Lichtes bei seinem Vorbeigehen an der Sonnennahe; in beiden Fallen ist die Ablenkung vom Massfelde abhängig.

3. Vorversuch betreffs der elektromagnetischen Feldgleichungen, die innerhalb isotroper Materie gelten mögen. Wir fragen nun: Wie gestalten sich die elektromagnetischen Feldgleichungen im Massfelde (1) bzw. (2). Hierzu verwenden wir die verallgemeinerten Feldgleichungen von Lorentz, wie sie bekannt sind, nämlich,

$$\left.\begin{aligned} &\frac{1}{\sqrt{-g}} \frac{\partial(\sqrt{-g}F^{\mu\nu})}{\partial x^\nu} = \pi\varrho \frac{dx^\mu}{ds} && (1) \\ &\frac{\partial F_{\mu\nu}}{\partial x^\sigma} + \frac{\partial F_{\nu\sigma}}{\partial x^\mu} + \frac{\partial F_{\sigma\mu}}{\partial x^\nu} = 0 && (2) \end{aligned}\right\} \tag{8}$$

und nicht die von Minkowski. Wir sind doch einstweilen dahin bestrebt diejenigen Eigenschaften der Materie an den Tag zu bringen, die möglicherweise in der Massformel stecken. Zieht man nun die bekannten Entzifferungsregeln, nämlich,

$$\left.\begin{aligned} (F_{23}, F_{31}, F_{12}) &= (B_x, B_y, B_z) = \vec{B} \\ (F_{11}, F_{24}, F_{34}) &= (E_x, E_y, E_z) = \vec{E} \end{aligned}\right\} \tag{9}$$

heran, so erhält man für das Massfeld (2) Feldgleichungen von der Form

$$\left.\begin{aligned} \operatorname{div} \vec{E} &= 4\pi\varrho \frac{h^{3/2}}{\sqrt{1-v^2/c^2}} \\ \operatorname{rot} \vec{B} &= 4\pi\varrho \frac{h^{3/2}}{\sqrt{1-v^2/c^2}} \left(\frac{dx}{cdt}, \frac{dy}{cdt}, \frac{dz}{cdt}\right) \end{aligned}\right\} \tag{8,1)$_2$ (}$$

bzw.

$$\left.\begin{aligned} \operatorname{div} \vec{B} &= 0 \\ \operatorname{rot} \vec{E} &= -\frac{\partial \vec{B}}{c\partial t} \end{aligned}\right\} \tag{8,2)$_2$ (}$$

in Abhängigkeit von den praktisch gebräuchlichen Vektoren $\vec{E}$, $\vec{B}$. Sie lauten fast genau gleich mit den bekannten Lorentz'schen Gleichungen für das Vakuum ausser dem Umstand, dass in $(8,1)_2$ rechter Hand $h^{3/2}$ in Verbindung mit ϱ auftritt. Für den Fall: $\varrho = 0$, ist aus $(8,1)_2$ und $(8,2)_2$ leicht ein Paar Wellengleichungen zu erhalten, wonach die Fortpflanzungsgeschwindigkeit des Lichtes sich zu c ergibt.

Verbindet man nun die Gleichungen (8) mit dem Massfelde (1), so erhält man mit Hilfe der gleichen Entzifferüngsregeln (9), wie oben, und in Abhängigkeit von den praktischen Vektoren, $\vec{E}$, $\vec{B}$, Feldgleichungen von der Form

$$\left.\begin{aligned} \operatorname{div}(n\vec{E}) &= 4\pi\varrho \frac{(fn)^{3/2}}{\sqrt{1-n^2v^2/c^2}} \\ \operatorname{rot}\left(\frac{\vec{B}}{n}\right) &= \frac{\partial(n\vec{E})}{c\partial t} + 4\pi\varrho \frac{(fn)^{3/2}}{\sqrt{1-n^2v^2/c^2}}\left(\frac{dx}{cdt}, \frac{dy}{cdt}, \frac{dz}{cdt}\right) \end{aligned}\right\} \quad (8,1)_1$$

bzw.

$$\left.\begin{aligned} \operatorname{div}\vec{B} &= 0 \\ \operatorname{rot}\vec{E} &= -\frac{\vec{B}}{cdt} \end{aligned}\right\} \quad (8.2)_1$$

Ist $\varrho = 0$ gesetzt, so folgert man leicht aus $(8,1)_1$ und $(8,2)_1$ Wellengleichungen von der Gestalt

$$\left.\begin{aligned} \frac{n^2}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} - \nabla^2\vec{E} &= 0 \\ \text{bzw.} \quad \frac{n^2}{c^2}\frac{\partial^2\vec{B}}{\partial t^2} - \nabla^2\vec{B} &= 0 \end{aligned}\right\} \quad (10)$$

woraus sich die Fortpflanzungsgechwindigkeit der Wellen zu c/n ergibt. Man hat also das befriedigende Resultat zu begrüssen, dass in den beiden Fällen das Massfeld und das in ihm eingebettete elektromanetische Feld dieselbe Lichtgeschwindigkeit liefern.

Wir wollen aber nach den Fall bewegter Materie untersuchen, dazu also die Gleichungen (8) mit dem vereinfachten Massfelde (4), d.h.

$$ds^2 = f'n.\left\{\frac{c^2dt^2}{n^2} + 2\left(1-\frac{1}{n^2}\right)dxudt - dx^2 - dy^2 - dz^2\right\}$$

in Verbindung* setzen. Um unnötige Erschwerung der Ausdrücke zu vermeiden wollen wir die diesbezügliche Rechnungen nur bis auf die erste Potenz von u/c treiben. Mit Hilfe von (9) liefern die Feldgleichungen (8) dann

* Siehe Anhang.

$$\left.\begin{aligned}
&\frac{\partial}{\partial x}\{nE_x\}+\frac{\partial}{\partial y}\left\{nE_y-(n^2-1)\frac{u}{c}\frac{B_z}{n}\right\}+\frac{\partial}{\partial z}\left\{nE_z+(n^2-1)\frac{u}{c}\frac{B_y}{n}\right\}\\
&\qquad\qquad=4\pi\varrho(f'n)^{3/2}.\left\{1+2(n^2-1)\frac{u}{c}\frac{v_x}{c}\right\}^{-\frac{1}{2}},\\
&\frac{\partial}{\partial y}\left\{\frac{B_z}{n}+\left(1-\frac{1}{n^2}\right)\frac{u}{c}nE_y\right\}-\frac{\partial}{\partial z}\left\{\frac{B_y}{n}-\left(1-\frac{1}{n^2}\right)\frac{u}{c}nE_z\right\}\\
&\qquad\qquad=\frac{\partial(nE_x)}{c\partial t}+4\pi\varrho(f'n)^{3/2}.\{1+2(n^2-1)\frac{u}{c}\frac{v_x}{c}\}^{-\frac{1}{2}}\cdot\frac{dx}{cdt},\\
&\frac{\partial}{\partial z}\left\{\frac{B_x}{n}\right\}-\frac{\partial}{\partial x}\left\{\frac{B_z}{n}+\left(1-\frac{1}{n^2}\right)\frac{u}{c}nE_y\right\}=\frac{\partial}{c\partial t}\left\{nE_y-(n^2-1)\frac{u}{c}\frac{B_z}{n}\right\}\\
&\qquad\qquad+4\pi\varrho(f'n)^{3/2}.\left\{1+2(n^2-1)\frac{u}{c}\frac{v_x}{c}\right\}^{-\frac{1}{2}}\cdot\frac{dy}{cdt},\\
&\frac{\partial}{\partial x}\left\{\frac{B_y}{n}-\left(1-\frac{1}{n^2}\right)\frac{u}{c}nE_z\right\}-\frac{\partial}{\partial y}\left\{\frac{B_x}{n}\right\}=\frac{\partial}{c\partial t}\left\{nE_z+(n^2-1)\frac{u}{c}\frac{B_y}{n}\right\}\\
&\qquad\qquad+4\pi\varrho(f'n)^{3/2}.\left\{1+2(n^2-1)\frac{u}{c}\frac{v_x}{c}\right\}^{-\frac{1}{2}}\cdot\frac{dz}{cdt}
\end{aligned}\right\}\quad(8,1)_4$$

bzw.

$$\left.\begin{aligned}
&\operatorname{div}\vec{B}=0\\
&\operatorname{rot}\vec{E}=-\frac{\partial\vec{B}}{c\partial t}
\end{aligned}\right\}\quad(8,2)_4$$

Falls keine Ladung vorliegt, so ist aus $(8, 1)_4$ und $(8, 2)_4$ das Paar Wellengleichungen zu erhalten von der Gestalt

$$\left.\begin{aligned}
&\frac{n^2}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}+2(n^2-1)\frac{u}{c}\frac{\partial^2\vec{E}}{c\partial t\partial x}-\nabla^2\vec{E}=0,\\
&\frac{n^2}{c^2}\frac{\partial^2\vec{B}}{\partial t^2}+2(n^2-1)\frac{u}{c}\frac{\partial^2\vec{B}}{c\partial t\partial x}-\nabla^2\vec{B}=0,
\end{aligned}\right\}\quad(11)$$

wenn man dabei die Grenze des Massfeldes ausschliesst und sich bei den Rechungen nur bis auf die erste Potenz von u/c beschränkt.

Mit dem Ansatz
$$(\vec{E},\vec{B})=(\vec{E}_0,\vec{B}_0)F\left(t-\frac{lx+my+nz}{w}\right)\quad(12)$$

erhält man aus (11) für die Bestimmung der Fortpflanzungsgeschwindigkeit w ebener Wellen die Gleichung

$$w^2-2\left(1-\frac{1}{n^2}\right)ulw-\frac{c^2}{n^2}=0\quad(13)$$

Mit $l = \cos\theta$ ergibt sich hieraus

$$w = \frac{c}{n} + \left(1 - \frac{1}{n^2}\right) u \cos\theta,$$

also der gleiche Ausdruck des Fresnel'schen Mitführungskoeffizientes mit dem bereits aus der Massformel (4) unmittelbar abgeleiteten.

Verbindet man die Feldgleichungen (8) mit der Massformel (5), so ergeben sich die gleichen Formen $(8, 1)_2$, $(8, 2)_2$, wie mit der Massformel (2), ausser dem Umstand, dass h' an Stelle von h erscheint. Das zugehörige Wellengleichungspaar liefert für die Fortpflanzungsgeschwindigkeit denselben Wert c.

Die wichtigsten Ergebnisse obenstehender Erörterungen sind nun, wie folgt, ins Wort zu fassen. Die tensoriellen Feldgleichungen (8) ergeben in Verbindung mit der Massformel (2) bzw. (5) stets die Lorentz'schen Feldgleichungen für das Vakuum, eventuell mit $h = 1 = h'$. In Verbindung mit den Massformeln (1) bzw. (4) liefern sie aber Gleichungssysteme, die eher demjenigen von Maxwell-Hertz ähnlich beschaffen zu sein scheinen—ähnlich, insofern sie die gleiche Lichtgeschwindigkeit innerhalb ruhender bzw. bewegter (isotroper) Materie ergeben wie die von Maxwell-Hertz bzw. Minkowski, jedoch davon verschieden, indem die Gleichungen (8, 1) auf Grenzbedingungen führen, welche von den aus den Gleichungen von Maxwell-Hertz bzw. Minkowski zu ableitenden wesentlich abweichen.

4. Die allgemeinen Feldgesetze des Elektromagnetismus, welche innerhalb der Materie als vollwertig zu erkennen sind. Die im Massfelde sich kundgebenden materiellen Eigenschaften geben uns an hand der Feldgleichungen (8) stets den richtigen Brechungsindex für elektromagnetische Wellen, vermögen aber nicht das volle elektromagnetische Gleichungsschema zu liefern. Jedoch glauben wir die tensoriellen elektromagnetischen Gesetze in folgender Gestalt formulieren und sie für das innere der Materie als vollwertig beanspruchen zu können. Von Minkowski abweichend schreiben wir die Gesetze wie

$$\left.\begin{aligned} \frac{1}{\sqrt{-g}} \frac{\partial\{\sqrt{-g}.\eta.F^{\mu\nu}\}}{\partial x^\nu} &= 4\pi\varrho\,\frac{dx^\alpha}{ds} \quad \ldots (1) \\ \frac{\partial F_{\mu\nu}}{dx^\sigma} + \frac{\partial F_{\nu\sigma}}{\partial x^\mu} + \frac{\partial F_{\sigma\mu}}{\partial x^\nu} &= 0 \;\ldots \quad \ldots (2) \end{aligned}\right\} \qquad (14)$$

η ist dabei als ein Lorentzinvarianter, dimensionsfreier Skalar zu verstehen, der an der Materie haften möge, im Vakuum aber gleich Eins sei.

Zieht man nun die Entzifferungsregeln (9) heran und setzt noch die nachstehenden hinzu, nämlich,

$$\left.\begin{aligned} \eta\sqrt{-g}(F^{23}, F^{31}, F^{12}) &= (H_x, H_y, H_z) = \vec{H} \\ \eta\sqrt{-g}(F^{41}, F^{42}, F^{43}) &= (D_x, D_y, D_z) = \vec{D} \end{aligned}\right\} \qquad (15)$$

So erhält man an Stelle von (14) die nachstehenden Gleichungen

$$\left.\begin{array}{l}\left.\begin{array}{l}\operatorname{div}\vec{D} = 4\pi\varrho\sqrt{-g}\;\dfrac{dx^4}{dx} \\[2ex] \operatorname{rot}\vec{H} = \dfrac{\partial\vec{D}}{c\partial t} + 4\pi\varrho\sqrt{-g}\left(\dfrac{dx^1}{ds},\ \dfrac{dx^2}{ds},\ \dfrac{dx^3}{ds}\right)\end{array}\right\}\ (1) \\[4ex] \left.\begin{array}{l}\operatorname{div}\vec{B} = 0 \\[2ex] \operatorname{rot}\vec{E} = -\dfrac{\partial\vec{B}}{c\partial t}\end{array}\right\}\ (2)\end{array}\right\}\quad(16)$$

welche die praktisch gebrauchten Vektorpaare $(\vec{D},\ \vec{H})$ bzw. $(\vec{E},\ \vec{B})$ enthalten.

Will man nun elektromagnetische Vorgänge in isotroper Materie bei Ruhe bzw. Bewegung verstehen, so hat man die obigen Gleichungen (14) oder (16) mit der Massformel (1) bzw. (4) in Verbindung zu bringen.

Fall der Ruhe [*die Massformel* (*1*)]. An $(8,\ 1)_1$ sieht man

$$\left.\begin{array}{l}\sqrt{-g}\,(F^{41},\ F^{42},\ F^{43}) = n\vec{E} \\[2ex] \sqrt{-g}\,(F^{23},\ F^{31},\ F^{12}) = \dfrac{1}{n}\vec{B}\end{array}\right\}\quad(17)$$

Im vorliegenden Falle hat man also nach (14) die elektromagnetischen Feldgleichungen der Form

$$\left.\begin{array}{l}\operatorname{div}(n\eta\vec{E}) = 4\pi\varrho\,\dfrac{(fn)^{3/2}}{\sqrt{1-n^2v^2/c^2}} \\[3ex] \operatorname{rot}\left(\dfrac{\eta\vec{B}}{n}\right) = \dfrac{\partial(n\eta\vec{E})}{c\partial t} + 4\pi\varrho\,\dfrac{(fn)^{3/2}}{\sqrt{1-n^2v^2/c^2}}\left(\dfrac{dx}{cdt},\ \dfrac{dy}{cdt},\ \dfrac{dz}{cdt}\right)\end{array}\right\}\quad(14,\ 1)_1$$

$$\left.\begin{array}{l}\operatorname{div}(\vec{B}) = 0 \\[2ex] \operatorname{rot}(\vec{E}) = -\dfrac{\partial\vec{B}}{c\partial t}\end{array}\right\}\quad(14,\ 2)_1$$

Die Äquivalenz dieser Gestalt mit der von Maxwell-Hertz ist sofort erkenntlich, falls man, wie wir es tun wollen, setzt

$$\left.\begin{array}{l}n\eta = \varepsilon\ [\text{Dielektrizitätskonstante}] \\[1ex] n/\eta = \mu\ [\text{Permeabilität}]\end{array}\right\}\quad(18)$$

Nach (15) gilt also übrigens

$$\left.\begin{aligned}\vec{D} &= \varepsilon\vec{E}\\ \vec{H} &= (1/\mu)\vec{B}\end{aligned}\right\} \tag{19}$$

Fall der Bewegung in der x-Richtung [*die Massformel* (4)]. An $(8, 1)_4$ sieht man nun

$$\left.\begin{aligned}\sqrt{-g}\; F^{41} &= nE_x\\ \sqrt{-g}\; F^{42} &= nE_y-(n^2-1)\frac{u}{c}\frac{B_z}{n}\\ \sqrt{-g}\; F^{43} &= nE_z+(n^2-1)\frac{u}{c}\frac{B_y}{n}\end{aligned}\right\} \tag{20, 1}$$

und

$$\left.\begin{aligned}\sqrt{-g}\; F^{23} &= \frac{1}{n}B_x\\ \sqrt{-g}\; F^{31} &= \frac{1}{n}B_y-\left(1-\frac{1}{n^2}\right)\frac{u}{c}\,nE_z\\ \sqrt{-g}\; F^{12} &= \frac{1}{n}B_z+\left(1-\frac{1}{n^2}\right)\frac{u}{c}\,nE_y\end{aligned}\right\} \tag{20, 2}$$

In diesem Falle erhält man also nach (14) die elektromagnetischen Feldgleichungen von der Gestalt

$$\left.\begin{aligned}&\frac{\partial}{\partial x}[\eta\{nE_x\}]+\frac{\partial}{\partial y}\left[\eta\left\{nE_y-(n^2-1)\frac{u}{c}\frac{B_z}{n}\right\}\right]+\frac{\partial}{\partial z}\left[\eta\left\{nE_z+(n^2-1)\frac{u}{c}\frac{B_y}{n}\right\}\right]\\ &\qquad\qquad= 4\pi\varrho(f'n)^{3/2}\left\{1+2(n^2-1)\frac{uv_x}{c^2}\right\}^{-\frac{1}{2}}\\ &\frac{\partial}{\partial y}\left[\eta\left\{\frac{B_z}{n}+\left(1-\frac{1}{n^2}\right)\frac{u}{c}\,nE_y\right\}\right]-\frac{\partial}{\partial z}\left[\eta\left\{\frac{B_y}{n}-\left(1-\frac{1}{n^2}\right)\frac{u}{c}\,nE_z\right\}\right]\\ &\qquad\qquad= \frac{\partial}{c\partial t}[\eta\{nE_x\}]+4\pi\varrho(f'n)^{3/2}\left\{1+2(n^2-1)\frac{uv_x}{c^2}\right\}^{-\frac{1}{2}}\frac{dx}{cdt}\\ &\frac{\partial}{\partial z}\left[\eta\left\{\frac{B_x}{n}\right\}\right]-\frac{\partial}{\partial x}\left[\eta\left\{\frac{B_z}{n}+\left(1-\frac{1}{n^2}\right)\frac{u}{c}\,nE_y\right\}\right]\\ &\qquad= \frac{\partial}{c\partial t}\left[\eta\left\{nE_y-(n^2-1)\frac{u}{c}\frac{B_z}{n}\right\}\right]+4\pi\varrho(f'n)^{3/2}\left\{1+2(n^2-1)\frac{uv_x}{c^2}\right\}^{-\frac{1}{2}}\frac{dy}{cdt}\\ &\frac{\partial}{\partial x}\left[\eta\left\{\frac{B_y}{n}-\left(1-\frac{1}{n^2}\right)\frac{u}{c}\,nE_z\right\}\right]-\frac{\partial}{\partial y}\left[\eta\left\{\frac{B_x}{n}\right\}\right]\\ &\qquad= \frac{\partial}{c\partial t}\left[\eta\left\{nE_z+(n^2-1)\frac{u}{c}\frac{B_y}{n}\right\}\right]+4\pi\varrho(f'n)^{3/2}\left\{1+2(n^2-1)\frac{uv_x}{c^2}\right\}^{-\frac{1}{2}}\frac{dz}{cdt}\end{aligned}\right\} \tag{14, 1}_4$$

$$\left.\begin{aligned} \operatorname{div} \vec{B} &= 0 \\ \operatorname{rot} \vec{E} &= -\frac{\partial \vec{B}}{c\partial t} \end{aligned}\right\} \qquad (14,\,2)_4$$

Wegen unserer Forderung der Lorentzinvarianz besitzt der Skalar η den gleichen Wert bei Bewegung, den er bei Ruhe hat, hier, nämlich $\sqrt{(\varepsilon/\mu)}$. Wir dürfen uns also der Setzungen (18) ohne Bedenken bedienen und in $(14,\,1)_4$ mit Rücksicht auf (15) setzen

$$\left.\begin{aligned} &\left.\begin{aligned} \eta(nE_x) &= \varepsilon E_x = D_x \\ \eta\left\{nE_y - (n^2-1)\frac{u}{c}\frac{B_z}{n}\right\} &= \varepsilon E_y - (n^2-1)\frac{u}{c}\frac{B_z}{\mu} = D_y \\ \eta\left\{nE_z + (n^2-1)\frac{u}{c}\frac{B_y}{n}\right\} &= \varepsilon E_z + (n^2-1)\frac{u}{c}\frac{B_y}{\mu} = D_z \end{aligned}\right\} \\ &\left.\begin{aligned} \eta\left(\frac{B_x}{n}\right) &= \frac{B_x}{\mu} = H_x \\ \eta\left\{\frac{B_y}{n} - \left(1-\frac{1}{n^2}\right)\frac{u}{c}\,nE_z\right\} &= \frac{B_y}{\mu} - \left(1-\frac{1}{n^2}\right)\frac{u}{c}\,\varepsilon E_z = H_y \\ \eta\left\{\frac{B_z}{n} + \left(1-\frac{1}{n^2}\right)\frac{u}{c}\,nE_y\right\} &= \frac{B_z}{\mu} + \left(1-\frac{1}{n^2}\right)\frac{u}{c}\,\varepsilon E_y = H_z \end{aligned}\right\} \end{aligned}\right\} \qquad (21)$$

Die Gegenüberstellung mit der Elektromagnetik von Minkowski für bewegte Medien.

Die Feldgesetze von Minkowski (Laue, 1921a) schreibt man bekanntlich, wie nachstehend, hin in fast der gleiche Form wie in (16)

$$\left.\begin{aligned} &\left.\begin{aligned} \operatorname{div} \vec{D} &= 4\pi\varrho \\ \operatorname{rot} \vec{H} &= \frac{\partial \vec{D}}{c\partial t} + 4\pi\frac{1}{c}(\vec{i} + \varrho\vec{q}) \end{aligned}\right\} \\ &\left.\begin{aligned} \operatorname{div} \vec{B} &= 0 \\ \operatorname{rot} \vec{E} &= -\frac{\partial \vec{B}}{c\partial t} \end{aligned}\right\} \end{aligned}\right\} \qquad (22)$$

daneben aber noch Verknüpfungsgleichungen wie

$$\left.\begin{aligned} \vec{D} + \frac{1}{c}[\vec{u}, \vec{H}] &= \varepsilon\left[\vec{E} + \frac{1}{c}[\vec{u}, \vec{B}]\right] \\ \vec{B} - \frac{1}{c}[\vec{u}, \vec{E}] &= \mu\left[\vec{H} - \frac{1}{c}[\vec{u}, \vec{D}]\right] \end{aligned}\right\} \tag{23}$$

Aus (23) folgert man

$$\left.\begin{aligned} \vec{D} &= \varepsilon\vec{E} + \frac{\varepsilon}{c}[\vec{u}, \vec{B}] - \frac{1}{c}[\vec{u}, \vec{H}] \\ &= \varepsilon\vec{E} + \frac{\varepsilon}{c}[\vec{u}, \vec{B}] - \frac{1}{c\mu}[\vec{u}, \vec{B}] \text{ in erster Naherung} \\ &= \varepsilon\vec{E} + \frac{\varepsilon\mu - 1}{c\mu}[\vec{u}, \vec{B}] \\ \vec{H} &= \frac{\vec{B}}{\mu} - \frac{1}{c\mu}[\vec{u}, \vec{E}] + \frac{1}{c}[\vec{u}, \vec{D}] \\ &= \frac{\vec{B}}{\mu} - \frac{1}{c\mu}[\vec{u}, \vec{E}] + \frac{\varepsilon}{c}[\vec{u}, \vec{E}] \text{ in erster Näherung} \\ &= \frac{\vec{B}}{\mu} + \frac{\varepsilon\mu - 1}{\varepsilon\mu} \cdot \frac{\varepsilon}{c}[\vec{u}, \vec{E}] \end{aligned}\right\} \tag{24}$$

Im von uns betrachteten Falle findet die Bewegung aber in der x-Richtung statt; daher ist zu setzen

$$\vec{u} = (u_x = u,\ u_y = 0,\ u_z = 0) \tag{25}$$

Demnach hat man aus (24)

$$\left.\begin{aligned} D_x &= \varepsilon E_x \\ D_y &= \varepsilon E_y - \frac{\varepsilon\mu - 1}{c\mu} u B_z \\ D_z &= \varepsilon E_z + \frac{\varepsilon\mu - 1}{c\mu} u B_y \end{aligned}\right\} \quad \left.\begin{aligned} H_x &= \frac{B_x}{\mu} \\ H_y &= \frac{B_y}{\mu} - \left(1 - \frac{1}{\varepsilon\mu}\right)\frac{u}{c}\varepsilon E_z \\ H_z &= \frac{B_z}{\mu} + \left(1 - \frac{1}{\varepsilon\mu}\right)\frac{u}{c}\varepsilon E_y \end{aligned}\right\} \tag{26}$$

Man sieht, dass die Gleichungen (26) mit den Beziehungen (21) vollständig übereinstimmen, wenn man darin: $\varepsilon\mu = n^2$, setzt.

Damit ist der Nachweis vollbracht dafür, dass die von uns oben formulierten Feldgesetze des Elektromagnestismus mit denen von Minkowski vollkommen übereinstimmen im Falle gleichförmiger Bewegung bis auf die erste Potenz von u/c. Die erste

Potenz von u/c ist es aber, die den Genauigkeitsgrad der Messunssergebnisse bei den Versuchen von (i) Wilson, (ii) Röntgen-Eichenwald, (iii) Fizeau und dergleichen darstellt; also decken jene Gesetze auch unsere bisherigen Erfahrungen bezuglich bewegter Materie vollkommen. Von der Minkowski'schen Theorie ist die oben behauptete prinzipiell dadurch unterschieden, dass sie etwa aufdeckt, wie die Materie auf elektromagnetische Vorgänge einwirkt; ihre Rolle ist, nämlich, durch zwei Momente völlig beschrieben, erstens, das innerhalb der Materie geltende Massfeld und zweitens der Lorentzinvariante, an der Materie anhaftende Skalar η Nach unserem Gesichtspunkte ist also bei homogener, isotroper Materie den Konstanten, n und η, eine primäre Bedeutung beizumessen, den Konstanten, ε und μ, dagegen eine deduktive, also sekundäre. Der formale Unterschied ist der, dass hier eine grössere Ökonomie im Sinne Machs erzielt wird Man findet sich, nämlich, bei obiger Formulierung mit einem einzigen antisymmetrischen Tensor ab, gegenüber den zwei solchen von Minkowski. Ausserdem fallen die zwei Verknüpfungsgleichungen von Minkowski fort—ihre Rolle ist gewissermassen von dem innerhalb der Materie geltenden Massfeld übernommen, teils aber auch vom Skalar η.

5. Der Impulsenergietensor und dessen Dichte. Den gemischten Impulsenergietensor schreiben wir als

$$\left.\begin{aligned} E_\sigma^{\nu} &= \frac{1}{4\pi}\eta M_{\sigma}^{\nu} \\ \text{wobei}\qquad M_\sigma^{\nu} &= -F_{\sigma\kappa}F^{\nu\kappa}+\tfrac{1}{4}\delta_\sigma^{\nu}F_{\alpha\beta}F^{\alpha\beta} \end{aligned}\right\} \tag{27}$$

Die Komponenten der Dichte dieses Tensors lassen sich nun leicht mit Hilfe der Regeln (9) bzw. (15) in Abhängigkeit von den praktischen Vektorpaaren $(\vec{D},\vec{H})$ und $(\vec{E},\vec{B})$, wie folgen, ausdrücken

Berücksichtigt man zunächst den Ausdruck

$$\tfrac{1}{4}\eta\sqrt{-g}F_{\alpha\beta}F^{\alpha\beta} = \tfrac{1}{2}(\vec{B}\vec{H})-\tfrac{1}{2}(\vec{E}\vec{D}), \tag{28}$$

dann hat man bequem

$$\left.\begin{aligned} \sqrt{-g}E_1^{1} &= \frac{1}{4\pi}\{E_xD_x-\tfrac{1}{2}(\vec{E}\vec{D})+B_xH_x-\tfrac{1}{2}(\vec{B}\vec{H})\} \\ \sqrt{-g}E_1^{2} &= \frac{1}{4\pi}\{E_xD_y+B_yH_x\} \\ \sqrt{-g}E_1^{3} &= \frac{1}{4\pi}\{E_xD_z+B_zH_x\} \\ \sqrt{-g}E_1^{4} &= \frac{1}{4\pi}\{B_yD_z-B_zD_y\} = \frac{1}{4\pi}[\vec{B},\vec{D}]_x \end{aligned}\right\} \tag{29,1}$$

$$\left.\begin{aligned}
\sqrt{-g}\,E_2^1 &= \frac{1}{4\pi}\{E_y D_x + B_x H_y\} \\
\sqrt{-g}\,E_2^2 &= \frac{1}{4\pi}\{E_y D_y - \tfrac{1}{2}(\vec{E}\vec{D}) + B_y H_y - \tfrac{1}{2}(\vec{B}\vec{H})\} \\
\sqrt{-g}\,E_2^3 &= \frac{1}{4\pi}\{E_y D_z + B_z H_y\} \\
\sqrt{-g}\,E_2^4 &= \frac{1}{4\pi}\{B_z D_x - B_x D_z\} = \frac{1}{4\pi}[\vec{B}\vec{D}]_y
\end{aligned}\right\} \qquad (29{,}2)$$

$$\left.\begin{aligned}
\sqrt{-g}\,E_3^1 &= \frac{1}{4\pi}\{E_z D_x + B_x H_z\} \\
\sqrt{-g}\,E_3^2 &= \frac{1}{4\pi}\{E_z D_y + B_y H_z\} \\
\sqrt{-g}\,E_3^3 &= \frac{1}{4\pi}\{E_z D_z - \tfrac{1}{2}(\vec{E}\vec{D}) + B_z H_z - \tfrac{1}{2}(\vec{B}\vec{H})\} \\
\sqrt{-g}\,E_3^4 &= \frac{1}{4\pi}\{B_x D_y - B_y D_x\} = \frac{1}{4\pi}[\vec{B}\vec{D}]_z
\end{aligned}\right\} \qquad (29{,}3)$$

$$\left.\begin{aligned}
\sqrt{-g}\,E_4^1 &= \frac{1}{4\pi}\{E_y H_z - E_z H_y\} = \frac{1}{4\pi}[\vec{E}\vec{H}]_x \\
\sqrt{-g}\,E_4^2 &= \qquad\qquad\qquad\qquad = \frac{1}{4\pi}[\vec{E}\vec{H}]_y \\
\sqrt{-g}\,E_4^3 &= \qquad\qquad\qquad\qquad = \frac{1}{4\pi}[\vec{E}\vec{H}]_z \\
\sqrt{-g}\,E_4^4 &= \frac{1}{4\pi}\{\tfrac{1}{2}(\vec{E}\vec{D}) + \tfrac{1}{2}(\vec{B}\vec{H})\}
\end{aligned}\right\} \qquad (29{,}4)$$

Man merkt, dass im allgemeinen

$$\sqrt{-g}\,E_2^1 \neq \sqrt{-g}\,E_1^2, \text{ usw.} \qquad (30)$$

Im Falle aber dass die Materie isotrop-homogen ist und dabei ruht, hat man die Formel (19) zur Verfügung und daher die Maxwell'schen Spannungen wieder in der üblichen symmetrischen Gestalt.

6. Die Viererkraft und ihre Beziehung zum Impulsenergietensor. Fur die Viererkraft, $\overset{\Lambda}{K}_\sigma$, setzen wir zunächst die übliche Formel

$$\overset{\Lambda}{K}_\sigma = \varrho F_{\sigma\mu}\frac{dx_\mu}{ds} \qquad (31)$$

und finden daher mittels (9)

$$\left.\begin{aligned}
\hat{K}_1 &= \varrho \frac{cdt}{ds}\left\{E_x + \frac{1}{c}[\vec{v}\vec{B}]_x\right\} \\
\hat{K}_2 &= \varrho \frac{cdt}{ds}\left\{E_y + \frac{1}{c}[\vec{v}\vec{B}]_y\right\} \\
\hat{K}_3 &= \varrho \frac{cdt}{ds}\left\{E_z + \frac{1}{c}[\vec{v}\vec{B}]_z\right\} \\
\hat{K}_4 &= -\varrho \frac{cdt}{ds}\,\frac{1}{c}(\vec{E}\vec{v})
\end{aligned}\right\} \tag{32}$$

Formt man nun die Kraftdichte, $\hat{K}_\sigma \sqrt{-g}$, mit Hilfe der Feldgleichungen (14) um, so findet man auf bekannte Weise aber

$$\begin{aligned}
\hat{K}_\sigma \sqrt{-g} &= \varrho \sqrt{-g}\,\frac{dx^\mu}{ds} F_{\sigma\mu} = \frac{1}{4\pi}\frac{\partial(\eta\sqrt{-g}\,F^{\mu\nu})}{\partial x^\nu} F_{\sigma\mu} \\
&= \frac{\partial}{\partial x^\nu}\left\{E_\sigma^\nu \sqrt{-g}\right\} - \tfrac{1}{2}\frac{\partial g_{sl}}{\partial x^\sigma} E^{sl}\sqrt{-g} \\
&\quad - \frac{1}{16\pi}\frac{\partial(\ln\eta)}{\partial x^\sigma}(\eta\sqrt{-g}\,F_{\alpha\beta}F^{\alpha\beta})
\end{aligned}$$

oder

$$\hat{K}_\sigma \sqrt{-g} + \sqrt{-g}\,\frac{\partial(\ln\eta)}{\partial x^\sigma}\left(\eta\,\frac{1}{16\pi}F_{\alpha\beta}F^{\alpha\beta}\right) = \frac{\partial}{\partial x^\nu}\left\{E_\sigma^\nu\sqrt{-g}\right\} - \tfrac{1}{2}\frac{\partial g_{sl}}{\partial x^\sigma}E^{sl}\sqrt{-g} \tag{33}$$

Den Ausdruck rechts in dieser Gleichung erkennt man als die Dichte der Divergenz des Impulsenergietensors E_σ^ν. Dagegen stellt der Zusatzausdruck links nämlich,

$$\sqrt{-g}\,\frac{d(\ln\eta)}{\partial x^\sigma}\left(\frac{\eta}{16\pi}F_{\alpha\beta}F^{\alpha\beta}\right) \tag{34}$$

ein den bisherigen Theorien fremdes, vermutlich aus der Wechselwirkung zwischen Materie und elektromagnetischem Felde stammendes Glied dar. Die Frage nach der wahren elektromagnetischen Kraftdichte wollen wir nun also dermassen entscheiden, dass auf dem Boden der Gravitionstheorie Einsteins die Erhaltungssatze der Impulse bzw. Energie erfüllt werden. Nach dieser Theorie lautet die Feldgleichung der Gravitation

$$I_\sigma^\nu - \tfrac{1}{2}\delta_\sigma^\nu R = -\kappa(T_\sigma^\nu + E_\sigma^\nu), \tag{35}$$

wobei T_σ^ν bzw. E_σ^ν der Impulsenergietensor der Materie bzw. des elektromagnetischen Feldes ist. Hieraus leitet man in bekannter Weise die Gleichung

$$\frac{\partial}{\partial x^\sigma}\{\sqrt{-g}\,(T^\nu_\sigma+E^\nu_\sigma+t^\nu_\sigma)\}=0 \tag{36}$$

her, welche die Erhaltung der gesamten Impulse bzw. Energie ausspricht. Hierbei ist, wie bekannt, t^ν_σ ein wegtransformierbarer Pseudotensor, welcher bei besonderer Wahl des Koordinatensystems [Z. B. bei geodätischen Koordinaten] wegfällt. Bei geodätischen Koordinaten hat man also nach (36)

$$-\frac{\partial}{\partial x^\nu}\left\{\sqrt{-g}\,T^\nu_\sigma\right\}=\frac{\partial}{\partial x^\nu}\left\{\sqrt{-g}\,E^\nu_\sigma\right\}=\overset{A}{K}_\sigma\sqrt{-g}+\sqrt{-g}\,\frac{\partial(\ln\eta)}{\partial x^\sigma}\left(\frac{\eta}{16\pi}F_{\alpha\beta}F^{\alpha\beta}\right) \tag{37}$$

gemäss (33). Die Dichte der ponderomotorischen etwa auf ein materielles Teilchen ausgeübten Kraft setzen wir also zu

$$\sqrt{-g}\left\{\overset{A}{K}_\sigma+\frac{\partial(\ln\eta)}{\partial x^\sigma}\left(\frac{\eta}{16\pi}F_{\alpha\beta}F^{\alpha\beta}\right)\right\} \tag{38}$$

in Abweichung von der üblichen Formel.

7. Die Elektromaganetik in erweiterter Fassung. Unsere frühere durch (14) gegebene Fassung der Elektromagnetik wollen wir nun probeweise durch

$$\left.\begin{aligned}&\frac{1}{\sqrt{-g}}\cdot\frac{\partial\{\sqrt{-g}(F^{\mu\nu}+4\pi p^{\mu\nu})\}}{\partial x^\nu}=4\pi\varrho\frac{dx^\mu}{ds}\\&\frac{\partial F_{\mu\nu}}{\partial x^\sigma}+\frac{\partial F_{\nu\sigma}}{\partial x^\mu}+\frac{\partial F_{\sigma\mu}}{\partial x^\nu}=0\end{aligned}\right\} \tag{39}$$

ersetzen, um dadurch möglicherweise dem physikalischen Sinn des von uns eingführten Lorentzinvinvarianten Skalars η auf die Spur kommen zu ku können. Setzt man

$$p^{\mu\nu}=\lambda F^{\mu\nu}, \tag{40}$$

wobei λ eine skalare, dimensionsfreie, in Vakuum aber verschwindende Grösse sei und weiterhin

$$1+4\pi\lambda=\eta, \tag{41}$$

so kommt man offenbar zur früheren Fassung zurück.

In Parallele mit dem antisymmetrischen Feldtensor $F_{\mu\nu}$ und angesichts der Regel (9) setzen wir

$$\left.\begin{aligned}(p_{23},p_{31},p_{12})&=-(M_x,M_y,M_z)=-\vec{M}\\(p_{14},p_{24},p_{34})&=-(P_x,P_y,P_z)=-\vec{P}\end{aligned}\right\} \tag{42}$$

und verstehen unter $\vec{M}$ bzw. $\vec{P}$ den Magnetisierungs—bzw. Polarisierungsvektor. Das negative Vorzeichen rechts in (42) wird darum gewählt, weil die Vektoren $\vec{M}$ bzw. $\vec{P}$, nach unserer Beurteilung, den entsprechenden Feldvektoren $\vec{B}$ bzw. $\vec{E}$ entgegengestzt

gerichtet seien. Bei Beschränkung, auf den Fall des ruhenden, isotropen Mediums ist dann nach (17) zu setzen

$$\left.\begin{aligned} \sqrt{-g}\,(p^{41}, p^{42}, p^{43}) &= -n\vec{P} \\ \sqrt{-g}\,(p^{23}, p^{31}, p^{12}) &= -\frac{1}{n}\vec{M} \end{aligned}\right\} \tag{43}$$

Demnach ist gemass (15) zu schreiben

$$\left.\begin{aligned} \vec{D} &= \sqrt{-g}\,(F^{41}+4\pi p^{41}, F^{42}+4\pi p^{42}, F^{43}+4\pi p^{13}) \\ &= n\vec{E}-4\pi n\vec{P} \\ \vec{H} &= \sqrt{-g}\,(F^{23}+4p^{23}, F^{31}+4\pi p^{31}, F^{12}+4\pi p^{12}) \\ &= \frac{\vec{B}}{n}-4\pi\frac{\vec{M}}{n} \end{aligned}\right\} \tag{44}$$

oder

$$\left.\begin{aligned} \vec{D} &= n\vec{E}-4\pi n\vec{P} \\ \vec{B} &= n\vec{H}+4\pi n\vec{M} \end{aligned}\right\} \tag{45}$$

statt

$$\left.\begin{aligned} \vec{D} &= \vec{F}+4\pi\vec{P} \\ \vec{B} &= \vec{H}+4\pi\vec{M} \end{aligned}\right\} \tag{46}$$

wie bisher nach Lorentz. Mit Rücksicht auf (40), (43), (45) und (41) hat man

$$\left.\begin{aligned} \vec{D} &= n\vec{E}+4\pi n\lambda\vec{E} = n\eta\vec{E} \\ \vec{B} &= n\vec{H}-4\pi\lambda\vec{B} \\ \text{d.h.}\quad \vec{B} &= \frac{n}{\eta}\vec{H} \end{aligned}\right\} \tag{47}$$

8. Schlussbetrachtungen. Am Schlusse wollen wir einige Fragen der Gravitationslehre kurz besprechen, die sich an die vorliegende Arbeit knüpfen. Wir sind, nämlich, von den Grundannahmen ausgegangen, dass die Massformel (1) in einem von Materie besetzten Raumgebiet herrscht, die Massformel (2) hingegen im umgebenden Vakuum. Wir gehen nun daran, jene Annahmen aus Verhalten der entsprechenden Krümmungsskalaren zu wägen. Schreibt man die Massformel (1), etwa, in der gleichwertigen Form

$$ds^2 = f\left[\frac{c^2dt^2}{n} - n\{dr^2+r^2d\theta^2+r^2\sin^2\theta d\varphi^2\}\right] \tag{48}$$

mit der Spezialisierung: (49)

$$f = f(r, t),$$

so berechnet man ohne allzugrosse Mühe den Krümmungsskalar als

$$\left.\begin{aligned} R = &-\frac{1}{nf}\left[3\left(\frac{f'}{f}\right)' + \frac{6}{r}\left(\frac{f'}{f}\right) + \frac{3}{2}\left(\frac{f'}{f}\right)^2\right] \\ &+ \frac{n}{f}\left[3\left(\frac{\mathring{f}}{f}\right)^{\circ} + \frac{3}{2}\left(\frac{\mathring{f}}{f}\right)^2\right], \end{aligned}\right\} \tag{50}$$

wobei

$$f' = \frac{\partial f}{\partial r}, \mathring{f} = \frac{\partial f}{c\partial t}, \text{ usw.,} \tag{51}$$

heissen. Für das Massfeld (1) ist also $R \neq 0$ und unsere diesbezügliche Annahme ist demnach mit der Gravitationslehre verträglich Schreibt man ebenfalls die Massformel (2) etwa wie

$$ds^2 = h(r,t)[c^2dt^2 - \{dr^2 + r^2d\theta^2 + r^2 \sin^2\theta d\varphi^2\}], \tag{52}$$

so findet man fur diesen Fall den Krummungsskalar als

$$\left.\begin{aligned} R = &-\frac{1}{h}\left[3\left(\frac{h'}{h}\right)' + \frac{6}{r}\left(\frac{h'}{h}\right) + \frac{3}{2}\left(\frac{h'}{h}\right)^2\right] \\ &+ \frac{1}{h}\left[3\left(\frac{\mathring{h}}{h}\right)^{\circ} + \frac{3}{2}\left(\frac{\mathring{h}}{h}\right)^2\right], \end{aligned}\right\} \tag{53}$$

wo $h' = \frac{\partial h}{\partial r}$, $\mathring{h} = \frac{\partial h}{c\partial t}$, usw., sind. Auch hier ist $R \neq 0$; nach der Gravitationslehre

zeigt sich also die Massformel (2) fur das Vakuum nicht hinreichend ohne Zusatzannahme wie etwa $h = 1$. An unsere Annahmen ist noch zu ersehen, dass gewisse Diskontinuitäten zwischen den Massfeldern (1) und (2) bestehen. Es scheint dies ein schroffer Verstoss gegen die Forderung der Kontinuitat zu sein, welche Gravitationstheoretiker bisher gemacht haben, so, zum Beispiel, zwischen dem Massfeld, welches innerhalb einer Flüssigkeitskugel (Laue, 1921*b*) herrscht und dem, welches im umgebenden Vakuum gilt. Diese Forderung beruht bekanntlich auf dem tensoriellen Satze

$$\frac{\partial\{\sqrt{-g}\,(T_n^m + t_n^m)\}}{\partial x^m} = 0 \tag{54}$$

oder im statischen Falle wenigstens dem algebraisch aquivalenten

$$\begin{aligned} &\frac{\partial}{\partial x^4}\int dx^1dx^2dx^3\{\sqrt{-g}(T_n^4 + t_n^4)\} \\ &= -\int dS\sqrt{-g}\,[a_1(T_n^1 + t_n^1) + a_2(T_n^2 + t_n^2) + a_3(T_n^3 + t_n^3)] \end{aligned} \tag{55}$$

Dabei sind a_1, a_2 a_3 die Richtungskosinusse der zum Flachestück dS der Begrenzung nach aussen gerichteten Normale Nach Einstein spricht dieser Satz die Erhaltung der

gessamten Energie bzw. Impulse aus. Zweifelsohne ist zuzugeben, dass in ihm etwas Objektives steckt wegen seines tensoriellen Charakters. Unberechtigt aber schliesst man hieraus auf die Kontinuität über die Grenzflächen von

$$\sqrt{-g}(\overset{1}{T}_n+\overset{1}{t}_n),\ \sqrt{-g}(\overset{2}{T}_n+\overset{2}{t}_n),\ \sqrt{-g}(\overset{3}{T}_n+\overset{3}{t}_n) \tag{56}$$

und infolge dessen auch von den Massfeldern. Auf Grund von (55) hätte man vielmehr auf folgende Weise vorgehen müssen: Schreibt man, nämlich, den Ausdruck rechts von (55), wie folgt, um

$$-\int dS\sqrt{-g}dn\,[a_1(\overset{1}{T}_n+\overset{1}{t}_n)+a_2(\overset{2}{T}_n+\overset{2}{t}_n)+a_3(\overset{3}{T}_n)]\frac{1}{dn} \tag{57}$$

wo dn das Element der Normale zu dS bezeichnet, so sieht man, dass

$$dS(\sqrt{-g}dn)_i = dS(\sqrt{-g}dn)_a$$

gelten muss, wobei $dS(\sqrt{-g}dn)_i$ bzw. $dS(\sqrt{-g}dn)_a$ den inneren bzw. äusseren Wert des dreidimensionalen Volumenelementes darstellt. Man hat daher vielmehr auf die Kontinuität über die Grenzflächen von

$$(\overset{1}{T}_n+\overset{1}{t}_n)\frac{1}{dn},\ \ (\overset{2}{T}_n+\overset{2}{t}_n)\frac{1}{dn},\ (\overset{3}{T}_n+\overset{3}{t}_n)\frac{1}{dn} \tag{58}$$

zu schliessen. Die Sache habe einmal dabei ihr Bewenden.

In diesem Zusammenhange wollen wir noch eine Frage aufwerfen, deren Beantwortung uns, physikalisch gesprochen, besonders wichtig erscheint. Mögen die Parameter-Koordinaten (x^4, x^1, x^2, x^3) grundsätzlich eine noch so willkürliche Benennung des Raumzeitpunktes sein, so gilt es doch zu wissen, wie man die zur Beobachtung geeigenten, rechtwinklig kartesischen Koordinaten zu wählen hat. Die Möglichkeit einer solchen Wahl bleibt eben nach der prinzipiellen Willkür der Parameter ganz offen. Ist der Beobachtungsort als Nullpunkt gewählt, so erkennen wir solche Koordinaten schlechthin als (t, x, y, z) in der Massformel

$$dS^2 = \left(1-\frac{\kappa}{4\pi}\int\frac{\sigma dV}{r}\right)c^2dt^2-\left(1+\frac{\kappa}{4\pi}\int\frac{\sigma dV}{r}\right)(dx^2+dy^2+dz^2), \tag{59}$$

welche schon von Einstein (1922) als Newton'sche Näherung gefasst worden ist. Am genauen Nullpunkte ist dann die Massformel eine Galiei'sche und in der nächsten Umgebung ist sie so beschaffen, dass der Beobachter das örtliche ihm gegenwärtige Gravitationsfeld daran wahrnimmt. Nachdem so an Hand der Newton'schen Näherung die Wahl der zur Beobachtung geeigneten rechtwinklig kartesisehen Koordinaten festgestellt wird, kann man das volle System der Gravitationsgleichungen Einsteins auch in diese Koordinaten ausdrücken, etwa, mit Setzungen wie

$$\left.\begin{aligned} g_{44} &= \left(1-\frac{\kappa}{4\pi}\int\frac{\sigma dV}{r}\right)+\gamma_{44}(t, x, y, z) \\ g_{11} &= -\left(1+\frac{\kappa}{4\pi}\int\frac{\sigma dV}{r}\right)+\gamma_{11}(t, x, y, z) \\ g_{14} &= \gamma_{14}(t, x, y, z), \\ g_{23} &= \gamma_{23}(t, x, y, z), \quad \text{usw.} \end{aligned}\right\} \tag{60}$$

Anhang. Nach der Massformel (4)

$$ds^2 = (f'n)\left[\frac{c^2dt^2}{n^2} + 2\left(1-\frac{1}{n^2}\right)udtdx - dx^2 - dy^2 - dz^2\right]$$

gelten:

$$g_{44} = \frac{f'}{n},\ g_{41} = (f'n)\left(1-\frac{1}{n^2}\right)\frac{u}{c},\ g_{11} = -(f'n) = g_{22} = g_{33},\ \text{sonst } g_{\mu\nu} = 0.$$

Ferner

$$g = \det\begin{vmatrix} \frac{f'}{n}, & (f'n)\left(1-\frac{1}{n^2}\right)\frac{u}{c}, & 0, & 0 \\ (f'n)\left(1-\frac{1}{n^2}\right)\frac{u}{c}, & -(f'n), & 0, & 0 \\ 0, & 0, & -(f'n), & 0 \\ 0, & 0, & 0, & -(f'n) \end{vmatrix} = -(f')^4n^2 \quad \text{bis auf}$$

die erste Potenz von $\frac{u}{c}$; $g^{44} = \frac{n}{f'}$, $g^{41} = \frac{n}{f'}\left(1-\frac{1}{n^2}\right)\frac{u}{c}$, $g^{11} = -\frac{1}{(f'n)} = g^{22} = g^{33}$, sonst $g^{\mu\nu} = 0$.

Ferner

$$\sqrt{-g}F^{41} = \sqrt{-g}\, g^{44}g^{11}F_{41} = nF_{14}.$$

$$\sqrt{-g}F^{42} = \sqrt{-g}\{g^{44}g^{22}F_{42} + g^{41}g^{22}F_{12}\} = nF_{24} - n\left(1-\frac{1}{n^2}\right)\frac{u}{c}F_{12},$$

$$\sqrt{-g}F^{43} = \sqrt{-g}\{g^{44}g^{33}F_{43} + g^{41}g^{33}F_{13} = nF_{34} + n\left(1-\frac{1}{n^2}\right)\frac{v}{c}F_{31},$$

$$\sqrt{-g}F^{23} = \sqrt{-g}\, g^{22}g^{33}F_{23} = \frac{1}{n}F_{23}$$

$$\sqrt{-g}F^{31} = \sqrt{-g}\{g^{33}g^{11}F_{31} + g^{33}g^{14}F_{34}\} = \frac{1}{n}F_{31} - n\left(1-\frac{1}{n^2}\right)\frac{u}{c}F_{34},$$

$$\sqrt{-g}F^{12} = \sqrt{-g}\{g^{11}g^{22}F_{12}+g^{14}g^{22}F_{42}\} = \frac{1}{n}F_{12}+n\left(1-\frac{1}{n^2}\right)\frac{u}{c}F_{24},$$

$$\sqrt{-g}\ \frac{cdt}{ds} = (f'n)^{3/2}\left\{1+2(n^2-1)\frac{uv_x}{c^4}-\frac{n^2v^2}{c^2}\right\}^{-\frac{1}{2}} = (f'n)^{3/2}\left\{1+2(n^2-1)\frac{uv_x}{c^4}\right\}^{-\frac{1}{2}}$$

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CALCUTTA.

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Einstein (1922) Vier Vorlesungen uber Relativitatstheorie, Princeton.
Laue, M v. (1921*a*) Die Relativitätstheorie Bd. 1.
——— (1921*b*) Die Relativitatstheorie Bd, 2.

TORSION OF COMPOSITE SECTIONS OF DIFFERENT ISOTROPIC MATERIALS

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(*Received—March 4, 1954.*)

1. In this paper function-theoretic method is adopted to solve the torsion problem of a cross-section which is composed of two or more different isotropic materials, when the cross section can be mapped conformally on the circle and the boundaries of regions of different materials are conformally represented on circumferences of concentric circles. Cases when such a cross-section contains a hole are also dealt with. Some of results in this paper agree with those of Payne (1949) obtained by a different method for corresponding problems.

2. The axis of the cylinder about which the twist τ is given is taken as the z-axis and the axes of x, y are taken in the plane of the cross-section of the cylinder. The cylinder is composed of an isotropic material of shear modulus μ_1 bounded by a curve C_1 within the cross-section which is bounded by the curve C_2 and also composed of a different isotropic material of shear modulus μ_2 occupying the space between C_1 and C_2.

Assuming the displacements along the co-ordinate axes to be

$$u_1 = -\tau yz,\ \ v_1 = \tau zx,\ \ w_1 = \tau\varphi_1 \text{ within } C_1,$$

and $$u_2 = -\tau yz,\ \ v_2 = \tau zx,\ \ w_2 = \tau\varphi_2 \text{ for space between } C_1 \text{ and } C_2, \tag{2.1}$$

the stress componenents become

$$\widehat{(xz)}_r = \tau\mu_r\left(\frac{\partial\varphi_r}{\partial x} - y\right),\ \ \widehat{(yz)}_r = \tau\mu_r\left(\frac{\partial\varphi_r}{\partial y} + y\right)$$

the suffix $r = 1$ is affixed for values within C_1 and $r = 2$ for those within C_1 and C_2

The stress equations of equilibrium

$$\frac{\partial\widehat{xz}}{\partial x} + \frac{\partial\widehat{yz}}{\partial y} = 0$$

shows that φ_1, φ_2 are harmonic within the regions they exist.

Choosing ψ_r conjugate to φ_r and putting

$$\Psi_r = \psi_r - \tfrac{1}{2}(x^2 + y^2) \qquad (r = 1, 2) \tag{2.2}$$

the above stress components become

$$\widehat{(xz)}_r = \tau\mu_r\frac{\partial\Psi_r}{\partial y},\ \ \widehat{(yz)}_r = -\tau\mu_r\frac{\partial\Psi_r}{\partial x} \qquad (r = 1, 2). \tag{2.3}$$

Since the displacements are continuous across the boundary C_1 we, from (2.1), have

$$\varphi_1 = \varphi_1 \text{ for points on } C_1 \tag{2.4}$$

Also since the boundary C_2 of the cylinder is free from tractions we have

$$\widehat{xz}\frac{dy}{ds} - \widehat{yz}\frac{dx}{ds} = 0 \text{ on } C_2$$

i.e.

$$\psi_2 = \tfrac{1}{2}(x^2 + y^2) \text{ on } C_2 \tag{2.5}$$

Since the tractions are continuous on boundary C_1 and hence from (2.3) we get

$$\mu_1 \Psi_1 = \mu_2 \Psi_2 \text{ on } \quad C_1.$$

It is possible to solve for φ_1, φ_2 from (2.4) – (2.6)

We now seek a function-theoretic solution. Let

$$z(= x + iy) = w(\sigma) = \sum_0^\infty a_n \sigma^n \tag{2.7}$$

transform conformally the region within C_1, and that between C_1 and C_2 on the region within γ_1, the circumference of a circle of radius r_1 and the region between γ_1 and γ_2, the circumference of a circle of radius r_2, respectively in the σ-plane. Then

$$\bar{z}z = \sum_{-\infty}^{\infty} b_n(r)\sigma^n$$

where

$$b_n(r) = \sum_{s=-\infty}^{\infty} a_{n+s}\bar{a}_s r^{2s}. \tag{2.8}$$

Assuming the complex torsion function in the σ-plane

$$F_1 = \varphi_1 + i\psi_1 = i\sum_0^\infty A_n \sigma^n, \quad F_2 = \varphi_2 + i\psi_2 = i\sum_{-\infty}^{\infty} B_n \sigma^n \tag{2.9}$$

we have by (2.4) – (2.6),

$$B_n - \frac{B_{-n}}{r_1^{2n}} = A_n, \tag{2.10}$$

$$B_n + \frac{B_{-n}}{r_2^{2n}} = b_n(r_2), \tag{2.11}$$

$$\mu_1\left\{B_n + \frac{B_{-n}}{r_1^{2n}} - b_n(r_1)\right\} = \mu_2\{A_n - b_n(r_1)\}. \tag{2.12}$$

Solving from above equations we have

$$A_n = \frac{(\mu_1 - \mu_2)r_1^{2n}b_n(r_1) - (\mu_1 + \mu_2)b_n(r_2)r_2^{2n} - r_2^{2n}(\mu_1 - \mu_2)\{b_n(r_2) - b_n(r_1)\}}{(\mu_1 - \mu_2)r_1^{2n} - (\mu_1 + \mu_2)r_2^{2n}}, \tag{2.13}$$

$$B_n = \frac{b_n(r_1)r_1^{2n}(\mu_1 - \mu_2) - b_n(r_2)r_2^{2n}(\mu_1 + \mu_2)}{(\mu_1 - \mu_2)r_1^{2n} - (\mu_1 + \mu_2)r_2^{2n}} \tag{2.14}$$

The torsional rigidity of the prism is

$$N = \mu_1 \int\int\left(x^2+y^2-x\frac{\partial\varphi_1}{\partial y}-y\frac{\partial\varphi_1}{\partial x}\right)dx\,dy+\mu_2\int\int\left(x^2+y^2-x\frac{\partial\varphi_2}{\partial y}-y\frac{\partial\varphi_2}{\partial x}\right)dx\,dy$$

where the first term above is integrated over the whole area bounded by C_1 and the second over the area included between C_1 and C_2. This, by transformation is

$$N = \mu_1\left\{-\tfrac{1}{4}i\int_{\gamma_1} w(\sigma)[w(1/\sigma)]^2 dw(\sigma)+\tfrac{1}{2}R\int_{\gamma_1} w(\sigma)w(1/\sigma)dF_1(\sigma)\right\}$$

$$+\mu_2\left[-\tfrac{1}{4}i\left\{\int_{\gamma_2} w(\sigma)[w(1/\sigma)]^2 dw(\sigma)-\int_{\gamma_1}\right\}\right.$$

$$\left.+\tfrac{1}{2}R\left\{\int_{\gamma_2} w(\sigma)w(1/\sigma)dF_2(\sigma)-\int_{\gamma_1}\right\}\right],$$

R standing before an integral meaning its real part.

Using Cauchy's Theorem of Residues,

$$N = \tfrac{1}{2}\pi\sum_{-\infty}^{\infty}[\mu_1 c_n(r_1)b_{-n}(r_1)+\mu_2\{c_n(r_2)b_{-n}(r_2)-c_n(r_1)b_{-n}(r_1)\}]-\mu_1\pi\sum_{1}^{\infty} nA_n b_{-n}(r_1)$$

$$-\mu_2\pi\sum_{-\infty}^{\infty} nB_n\{b_{-n}(r_2)-b_{-n}(r_1)\} \qquad (2.15)$$

where

$$c_n(r) = \sum_{s=-\infty}^{\infty}(n+s)a_{n+s}\bar{a}_s r^{2s} \qquad (2.16)$$

3. As an example, we take the cross-section in the form of a circle of radius r_2 of which a portion of circle of radius $r_1(r_1<r_2)$ is composed of material of shear modulus μ_1 and the portion included between circles of radii r_1 and r_2 with material of shear modulus μ_2.

In this case

$$z = w(\sigma) = \sigma$$

$$b_0 = r^2, \quad b_n = 0 \qquad (n\geq 1)$$

$$B_n = A_n = 0, \quad c_n = 0 \quad (n\geq 1), \quad c_0 = r^2$$

$$\therefore \qquad F_1 = F_2 = 0$$

and

$$N = \tfrac{1}{2}\pi[\mu_1 r_1^4+\mu_2(r_2^4-r_1^4)]$$

The result agrees with Payne (1949)

4. The above process may be adopted for a cross-section bounded by the curve C_k composed of different isotropic materials of shear moduli $\mu_1, \mu_2, \ldots, \mu_k$ filling up regions within boundaries C_1, between C_1 and C_2, between C_2 and C_3, , between C_{k-1}, C_k

Let

$$z = w(\sigma) = \sum_{0}^{\infty} a_n\sigma^n \qquad (4.1)$$

map the region in the z-plane bounded by C_k into a circle of radius r_k, other boundaries $C_1, C_2, \ldots\ldots$ corresponding to circles of radii $r_1, r_2, \ldots\ldots$ in the σ-plane. Then as in art. 2 choosing

$$\left.\begin{aligned} F_r &= \varphi_r + i\psi_r = \sum_{n=-\infty}^{\infty} A_{rn}\sigma^n \quad (r = 2, 3, \ldots, k-1) \\ \text{and} \qquad F_1 &= \sum_{0}^{\infty} A_{1n}\sigma^n \end{aligned}\right\} \tag{4.2}$$

we get analogous to (2.4), $k-1$ relations

$$\varphi_1 = \varphi_2 \text{ on } r = r_1, \quad \varphi_2 = \varphi_3 \text{ on } r = r_2, \ldots, \varphi_{k-1} = \varphi_k \text{ on } r = r_{k-1}$$

giving
$$A_{2n} - \frac{\bar{A}_{-2n}}{r_1^{2n}} = A_{1n}, \; A_{sn} - \frac{\bar{A}_{-sn}}{r_s^{2n}} = A_{(s+1),n} - \frac{\bar{A}_{-(s+1),n}}{r_s^{2n}} \tag{4.3}$$
$$s = 2, 3, \ldots, (k-1).$$

Also analogous to (2.6) we get $k-1$ relations

$$\mu_s \Psi_s = \mu_{s+1} \Psi_{s+1} \text{ on } r = r_s, \; [s = 1, 2, \ldots, (k-1)]$$

where
$$\Psi_s = \psi_s - \tfrac{1}{2}(x^2 + y^2),$$

giving
$$\mu_s\left\{A_{s,n} + \frac{\bar{A}_{-s,n}}{r_{s-1}^{2n}} - b_n(r_{s-1})\right\} = \mu_{s-1}\left\{A_{s-1,n} + \frac{\bar{A}_{-(s-1),n}}{r_{s-1}^{2n}} b_n(r_{s-1})\right\},$$
$$(s = 2, 3, \ldots, k) \tag{4.4}$$

Also since the bounding curve $r = r_k$ is free from tractions, we have

$$A_{kn} + \frac{\bar{A}_{-k,n}}{r_k^{2n}} = b_n(r_k) \tag{4.5}$$

The $(2k-1)$ equations in $(4.3)-(4.5)$ give the values of $(2k-1)$ constants $A_{1,n}; A_{2,n}; \bar{A}_{-2,n}; A_{3,n}; \bar{A}_{-3,n}; \ldots\ldots\ldots A_{k,n}, \bar{A}_{-k,n}$.

5. We now consider the cylinder having a hole bounded by a curve C_1. It is composed of an isotropic material of shear modulus μ_1 between C_1 and a closed curve C_2, and of material of shear modulus μ_2 between C_2 and C_3, the bounding curve of the cross-section.

Let
$$z = \sum_{-\infty}^{\infty} a_n \sigma_n \tag{5.1}$$

transform conformally the region between C_1 and C_2 and that between C_2 and C_3, in the z-plane into region bounded by circles of radii r_1 and r_2 and that bounded by circles of radii r_2 and r_3 respectively. $(r_1 < r_2 < r_3)$. Choosing the complex torsion function

$$\left.\begin{aligned} F_1 &= \sum_{-\infty}^{\infty} A_n \sigma^n, \quad r_1 \leq \sigma < r_2 \\ F_2 &= \sum_{-\infty}^{\infty} B_n \sigma^n \quad r_2 < \sigma \leq r_3 \end{aligned}\right\} \tag{5.2}$$

We get by conditions analogous to (2.4) and (2.6)

$$A_n - \frac{\bar{A}_{-n}}{r_2^{2n}} = B_n - \frac{\bar{B}_{-n}}{r_2^{2n}}, \tag{5.3}$$

$$\mu_1\left\{A_n + \frac{\bar{A}_{-n}}{r_2^{2n}} - b_n(r_2)\right\} = \mu_2\left\{B_n + \frac{\bar{B}_{-n}}{r_2^{2n}} - b_n(r_2)\right\}. \tag{5.4}$$

Also since the boundaries $\sigma = r_1$, and $\sigma = r_3$ are free from tractions we get

$$A_n + \frac{\bar{A}_{-n}}{r_1^{2n}} = b_n(r_1), \tag{5.5}$$

$$B_n + \frac{\bar{B}_{-n}}{r_3^{2n}} = b_n(r_3). \tag{5.6}$$

Solving from (5.3) – (5.6), we get

$$A_n = \frac{\mu_2(r_3^{2n} - r_2^{2n})\{b_n(r_1)r_1^{2n} - b_n(r_3)r_3^{2n}\} + (r_2^{2n} + r_3^{2n})[\mu_1\{r_2^{2n}b_n(r_2) - r_1^{2n}b_n(r_1)\} + \mu_2\{r_3^{2n}b_n(r_3) - r_2^{2n}b_n(r_2)\}]}{\mu_2(r_3^{2n} - r_2^{2n})(r_1^{2n} + r_2^{2n}) + \mu_1(r_2^{2n} - r_1^{2n})(r_2^{2n} + r_3^{2n})}, \tag{5.7}$$

$$B_n = \frac{(r_1^{2n} + r_2^{2n})[\mu_1\{r_2^{2n}b_n(r_2) - r_1^{2n}b_n(r_1)\} + \mu_2\{r_3^{2n}b_n(r_3) - r_2^{2n}b_n(r_2)\}] - \mu_1(r_2^{2n} - r_1^{2n})\{b_n(r_1)r_1^{2n} - b_n(r_3)r_3^{2n}\}}{\mu_2(r_3^{2n} - r_2^{2n})(r_1^{2n} + r_2^{2n}) + \mu_1(r_2^{2n} - r_1^{2n})(r_2^{2n} + r_3^{2n})} \tag{5.8}$$

Torsional rigidity is given by

$$N = \mu_1 \sum_{-\infty}^{\infty} [\tfrac{1}{2}\pi\{c_n(r_2)b_{-n}(r_2) - c_n(r_1)b_{-n}(r_1)\} - \pi n A_n\{b_{-n}(r_2) - b_{-n}(r_1)\}]$$

$$+ \mu_2 \sum_{\infty}^{\infty} [\tfrac{1}{2}\pi\{c_n(r_3)b_{-n}(r_3) - c_n(r_2)b_{-n}(r_2)\} - \pi n B_n\{b_{-n}(r_3) - b_{-n}(r_2)\}] \tag{5.9}$$

where c_n, b_n have usual meanings.

6. As an example of above section 5 we consider an elliptic hole within a confocal elliptic cross-section and materials are within region bounded by confocal ellipses *i.e.* curves C_1, C_2, C_3 of section 5 are confocal ellipses. The mapping function in this case is

$$z = w(\sigma) = c\left(\sigma + \frac{1}{\sigma}\right)$$

transforming the boundaries C_1, C_2, C_3 into circumferences of circles of radii r_1, r_2, r_3 in the σ-plane. We have

$$2cr_1 = a_1 + a_2,\ 2cr_2 = a_2 + b_2,\ 2cr_3 = a_3 + b_3,$$

$$4c^2 = a_1^2 - b_1^2 = a_2^2 - b_2^2 = a_3^2 - b_3^2$$

where a_1, b_1 etc. are semi-axes of ellipses.

In this case constants

$$a_1 = a_{-1} = c, \text{ all other } a_n = 0$$

$$b_2 = c^2 r^{-2}, \; b_{-2} = c^2 r^2 \text{ all other } b_n = 0$$

$$c_0 = c^2\left(r^2 + \frac{1}{r^2}\right), \; c_2 = c^2 r^{-2}, \; c_{-2} = c^2 r^2, \text{ all other } c_n = 0.$$

$$A_2 = c^2 \frac{\mu_2(r_3^4 - r_2^4)(r_1^2 - r_3^2) + (r_2^4 + r_3^4)[\mu_1(r_2^2 - r_1^2) + \mu_2(r_3^2 - r_2^2)]}{\mu_2(r_3^4 - r_2^4)(r_1^4 + r_2^4) + \mu_1(r_2^4 - r_1^4)(r_2^4 + r_3^4)}, \tag{6.1}$$

$$B_2 = c^2 \frac{(r_1^4 + r_2^4)\{\mu_1(r_2^2 - r_1^2) + \mu_2(r_3^2 - r_2^2)\} - \mu_1(r_2^4 - r_1^4)(r_1^2 - r_3^2)}{\mu_2(r_3^4 - r_2^4)(r_1^4 + r_2^4) + \mu_1(r_2^4 - r_1^4)(r_2^4 + r_3^4)} \tag{9.2}$$

$$A_{-2} = c^2 \frac{\mu_2(r_3^{-4} - r_2^{-4})(r_1^{-2} - r_3^{-2}) + (r_2^{-4} + r_3^{-4})\{\mu_1(r_2^{-2} - r_1^{-2}) + \mu_2(r_3^{-2} - r_2^{-2})\}}{\mu_2(r_3^{-4} - r_2^{-4})(r_1^{-4} + r_2^{-4}) + \mu_1(r_2^{-4} - r_1^{-4})(r_2^{-4} + r_3^{-4})} \tag{6.3}$$

$$B_{-2} = c^2 \frac{(r_1^{-4} + r_2^{-4})\{\mu_1(r_2^{-2} - r_1^{-2}) + \mu_2(r_3^{-2} - r_1^{-2})\} - \mu_1(r_2^{-4} - r_1^{-4})(r_1^{-2} - r_3^{-2})}{\mu_2(r_3^{-4} - r_2^{-4})(r_1^{-4} + r_2^{-4}) + \mu_1(r_2^{-4} - r_1^{-4})(r_2^{-4} + r_3^{-4})} \tag{6.4}$$

all other A_n, B_n are zeros

If we put $r_2 = r_3$, $\mu_1 = \mu_2$, we get the case of an isotropic cylinder whose cross-section is bounded by two confocal ellipses. The result agrees with that of Stevenson (1948).

7. As a second example we consider the cross-section bounded by eccentric circles

$$(x-h_r)^2 + y^2 = k_r^2 \qquad (r = 1, 3, r_3 > r_1)$$

and the material of shear modulus μ_1 is between

$$(x-h_1)^2 + y^2 = k_1^2 \text{ and } (x-h_2)^2 + y^2 = k_2^2$$

and that of shear modulus μ_2, between

$$(x-h_2)^2 + y^2 = k_2^2. \; (x-h_3)^2 + y^2 = k_3^2$$

Let the mapping formula

$$z = \frac{c}{1-\sigma} \tag{7.1}$$

transform the boundaries on circles of radii r_1, r_2, r_3 respectively and

$$h(r) = \frac{c}{1-r^2}, \quad k(r) = \frac{cr}{1-r^2}, \; r = \frac{k(r)}{h(r)} \text{ for } r = r_1, r_2, r_3. \tag{7.2}$$

In this case $\qquad a_n = c, \; n \geq 0 \text{ and } a_{-n} = 0.$ (7.3)

$$b_n = \frac{c^2}{1-r^2} = ch(r) \;\; n \geq 0; \; b_{-n} = r^{2n} c h(r) \;\; n \geq 0 \tag{7.4}$$

$$c_n = nch(r) + k^2, \; c_{-n} = k^{2n+2}(r)/h^2(r) \tag{7.5}$$

$$\therefore\ A_n = c^2 \frac{\mu_1(r_2^{2n}+r_3^{2n})\{p(r_2)-p(r_1)\}+\mu_2\{2r_2^{2n}p(r_3)-(r_2^{2n}+r_3^{2n})p(r_2)-(r_2^{2n}-r_3^{2n})p(r_1)\}}{\mu_1(r_2^{2n}-r_1^{2n})(r_2^{2n}+r_3^{2n})-\mu_2(r_1^{2n}+r_2^{2n})(r_2^{2n}-r_3^{2n})}$$

$$B_n = c^2 \frac{(r_1^{2n}+r_2^{2n})[\mu_1\{p(r_2)-p(r_1)\}+\mu_2\{p(r_3)-p(r_2)\}]-\mu_1(r_2^{2n}-r_1^{2n})\{p(r_1)-p(r_3)\}}{\mu_1(r_2^{2n}-r_1^{2n})(r_2^{2n}+r_3^{2n})-\mu_2(r_1^{2n}+r_2^{2n}(r_2^{2n}-r_3^{2n})}$$

where
$$p(r) = \frac{r^{2n}}{1-r^2}, \quad (r = r_1, r_2, r_3)$$

If we put above $r = r_3$, $\mu_1 = \mu_2$, the result agrees with that of Stevenson (1948) for a completely isotropic case of the uniform material.

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References

Payne, L. E., (1949), *Iowa State Coll. Jour. of Sc.* 23, 381.
Stevenson, A. C. (1948), *Phil. Mag.* 39, 297.

A NOTE ON CERTAIN PLANE SETS OF POINTS

By

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(*Received—April 9, 1954*)

In an interesting note Randolph (1940) proves that the distance set for Cantor's discontinuum takes up every value in the intewal $0 \leq x \leq 1$. W. R. Utz. (1951) gives a simpler proof of the above theorem.

Following the method used by Utz to prove Randolph's theorem; we construct in this note a plane set E of the first category and of measure zero contained in the unit square Q whose vertices are $(0. 0)$, $(1, 0)$, $(1, 1)$ and $(0, 1)$, such that any straight line $y = mx + c$, which is not parallel to either of the axes and which has a point common with Q, has at least one point common with E.

Let n be a positive integer. We divide the interval $0 \leq x \leq 1$ which we call I into $(2n+1)$ equal parts and suppress the open middle sub interval, viz, $n/(2n+1) < x < (n+1)/(2n+1)$ from I. There now remain two closed subintervals each of length $n/(2n+1)$ one to the left and the other to the right of the interval suppressed. We denote them by I_0 and I_1 respectively. Each of these subintervals is again divided into $(2n+1)$ equal parts and the open middle interval from each is suppressed. The length of each of these two suppressed parts is $n/(2n+1)^2$. There now remain four closed subintervals. Two of them lie in I_0. We call them I_{00}, I_{01} and two of them lie in I_1. We call them I_{10}, I_{11}. Each of these intervals is again treated in a similar manner. Four open subintervals each of length $n^2/(2n+1)^3$ are now suppressed, and two closed subintervals survive in each of the four closed subintervals I_{00}, I_{01}. I_{10}, I_{11}. The subintervals that survive in I_{00} may be denoted by I_{000}, I_{001}; those that survive in I_{01}, are denoted by I_{010}, I_{011} and so on. The process being supposed continued indefinitely an everywhere dense set of non-overlapping open intervals of measure unity will be eliminated from I.

Let us write

$$I_0 \cup I_1 = I^{(1)}$$

$$I_{00} \cup I_{01} \cup I_{10} \cup I_{11} = I^{(2)}$$

$$I_{000} \cup \ldots\ldots\ldots\ldots\ldots \cup I_{111} = I^{(3)}$$

$$\cdots \qquad \cdots \qquad \cdots \qquad \cdots \qquad \cdots$$

Evidently $I \supset I^{(1)} \supset I^{(2)} \supset I^{(3)} \supset \ldots\ldots\ldots$, and each $I^{(n)}$ is closed. Then inner limiting set $E'_n = I \cap I^{(1)} \cap I^{(2)} \cap \ldots\ldots\ldots$, is a non-dense perfect set of measure zero on $0 \leq x \leq 1$.

We put a similar set E''_n on the unit interval $0 \leq y \leq 1$ on the y-axis. Let E_n be the product set of E'_n and E''_n. E_n is thus the set of all points (x, y) of Q where $x \in E'_n$ and $y \in E''_n$.

It is easy to prove following Utz that given $m(\neq 0)$ such that $1/(2n+1)\leq|m|\leq 2n+1$, if the line $y=mx+c$ has a point common with Q, then it has at least one point common with E_n.

Giving to n the values 1, 2, 3,......successively we construct an enumerable sequence of non-dense perfect sets E_1, E_2, E_3.........all lying in Q. Consider the union $E = E_1 \cup E_2 \cup E_3 \cup$......... The set E is contained in Q and is of the first category. It is of measure zero.

Now, take a line, not parallel to either of the axes Its equation is $y = mx+c$, where $m \lesseqgtr 0$. Suppose, it has a point common with Q. It is possible to find a positive integer n such that $1/(2n+1)<|m|<2n+1$. So the line will have a point common with E_n and therefore common with E.

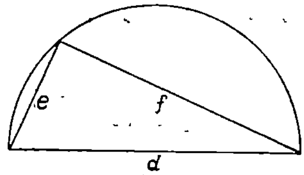
Note :—In order to shew that every set of the first category lying in Q, even if it be everywhere dense in Q, and has the power of the continuum does not necessarily possess the above property, we consider an everywhere dense set of points A lying in Q defined by (x, y), where x runs over all rational numbers in $0\leq x\leq 1$, while y runs over all irrational numbers in $0<y<1$. The plane set of points A is everywhere dense in Q and is of the first category.

Now take a line $y = mx+c$ where $m\neq 0$ and m and c are rational. Any point (α, β) on this line is such that either both of α and β are rational or both are irrational. So, even though the line may have points common with Q, it would not go through any point of A.

We now prove the following theorem.

Theorem. *For any set E_n as constructed above the distance set between points of the set completely fills the closed interval $0\leq x\leq\sqrt{2}$, and that to any d such that $0<d<\sqrt{2}$ there are an infinity of pairs of points of E_n such that the distance between points of each pair is precisely d.*

Proof. Firstly, let d lie in $0<x\leq 1$. It is possible to choose infinite pairs of numbers (e, f) such that $e^2+f^2 = d^2$. Here $0<e<d$; $0<f<d$, and e may take up every value between 0 and d, and e being chosen, f is determined by, $f = (d^2-e^2)^{\frac{1}{2}}$. Now d being given we take a fixed e such that $0<e<d$. Then there are two points by Randolph's Theorem on E'_n, $x^{(1)}_e$ and $x^{(2)}_e$ say, the distance between which is precisely e. We now take the corresponding $f = (d^2-e^2)^{\frac{1}{2}}$. On E''_n there are two points $y^{(1)}_f$ and $y^{(2)}_f$, the distance between which is precisely f.



Consider the points $(x^{(1)}_e, y^{(1)}_f)$, $(x^{(2)}_e, y^{(1)}_f)$, $(x^{(2)}_e, y^{(2)}_f)$ and $(x^{(1)}_e, y^{(2)}_f)$. These belong to E_n and form the corners of a rectangle, one of whose sides is e in length while the other

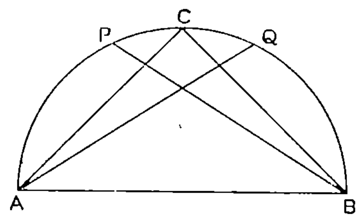
side is f. It follows that the diagonals of the rectangle are each d in length. The distance between the points $(x_e^{(1)}, y_f^{(1)})$ and $(x_e^{(2)}, y_f^{(2)})$ is d. Also the distance between the points $(x_e^{(1)}, y_f^{(2)})$ and $(x_e^{(2)}, y_f^{(1)})$ is d.

Now, take another e say e'. To this, there corresponds another f. We call it f', such that $e'^2+f'^2 = d^2$.

To e', correspond a pair of points on E'_n which we write as $x_e^{(1)'}$ and $x_e^{(2)'}$ such that the distance between them is e'. At least one of these points will be different from each of the points $x_e^{(1)}$ and $x_e^{(2)}$.

Thus proceeding as before, we get a rectangle distinct from the above, with corners on E_n such that its diagonals are each equal to d. We thus get a different pair of points of E_n, the distance between which is d.

As to a single d, the set of values that e may take has the power of the continuum, the number of rectangles that emerge with corners on E_n and having diagonals each of length d must be infinite. For the contrary assumption, forces us to the conclusion that the pairs of points $(x_e^{(1)}, x_e^{(2)})$ etc on E'_n having distance e between them, where $\{e\}$ has the power of the continuum is finite, which is impossible.



If $1<d<\sqrt{2}$, draw $AB = d$. If C be the mid-point of the semi-circle, $AC = BC<1$. We may therefore take two points P and Q on two sides of C such that $AQ = BP = 1$. We may therefore make e take infinitely many values between $d/\sqrt{2}$ and 1. and once e has been chosen, we take f such that $e^2+f^2 = d^2$. And the argument indicated above provides us with an infinity of rectangles with corners on E_n, each having a diagonal equal to d in length, where $1<d<\sqrt{2}$. If $d = \sqrt{2}$, there are two pairs of points (0, 0), (1, 1) and (0, 1), (1, 0) the distance between each of which is $\sqrt{2}$. This completes the proof.

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References

Randolph, J. F (1940), *Amer. Math Monthly*, **47**, 519.
Utz, W. R (1951). *Amer Math. Monthly*, **58**, 407.

ON THE CONVERGENCE CRITERION OF AN OSCILLATING SERIES

By

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(*Communicated by the Secretary—Received March 26, 1954*)

Suppose that $\varphi(t)$ is an even function, integrable in Lebesgue sense and having 2π as its period. Let its Fourier series be

(1)
$$\varphi(t) \approx \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nt.$$

Let
$$s_n = \tfrac{1}{2}a_0 + \sum_{\nu=1}^{n} a_\nu$$

be the n-th partial sum of the oscillating series $\sum a_n$. Write

$$\Phi(t) = \int_0^t \varphi(u)du$$

In this note, we shall establish a new convergence criterion* for the Fourier series (1) at the point $t = 0$. We prove the following

Theorem. *Let Δ be some non-negative value less than unity.*

If (i)
$$\varphi(t) = o\{(\log 1/t)^{-\Delta}\}$$
as $t \to +0$ and

(ii)
$$a_n > -K(\log n)^{\Delta}/n$$

for some positive value K, then the series $\sum a_n$ converges to the sum $s = 0$

Before establishing the theorem, we require the following lemmas.

Lemma 1. *Let σ_n be the first arithmetic means of the sequence $\{s_n\}$. If the condition* (i) *is satisfied, then $\sigma_n = o\{(\log n)^{-\Delta}\}$ as $n \to \infty$.*

Proof. We have

$$\sigma_n = \frac{1}{\pi(n+1)} \int_0^\pi \varphi(t)\left(\frac{\sin \frac{1}{2}(n+1)t}{\sin \frac{1}{2}t}\right)^2 dt = \frac{1}{\pi(n+1)} \left(\int_0^{n^{-\gamma}} + \int_{n^{-\gamma}}^{\pi}\right)$$

$$= \frac{1}{\pi(n+1)} (I_1 + I_2)$$

* For further references concerning the devolopment of the convergence criterion of this type, cf. Hardy and Rogosinski, (1946). pp. 45 and 96.

where $0<\gamma<\frac{1}{2}$ being chosen previously. Now,

$$|I_1| \leqq \int_0^{n^{-\gamma}} |\varphi(t)| \left(\frac{\sin\frac{1}{2}(n+1)t}{\sin\frac{1}{2}t}\right)^2 dt \leqq \max_{0<t<n^{-\gamma}} |\varphi(t)| \int_0^{\pi} \left(\frac{\sin\frac{1}{2}(n+1)t}{\sin\frac{1}{2}t}\right)^2 dt$$

$$= \pi(n+1) \max_{0<t<n^{-\gamma}} |\varphi(t)| = o\left(\frac{n}{(\log n)^{\Delta}}\right)$$

by (i), since

$$\frac{1}{\pi(n+1)} \int_0^{\pi} \left(\frac{\sin\frac{1}{2}(n+1)t}{\sin\frac{1}{2}t}\right)^2 dt = 1.$$

$$I_2 = 2\int_{n^{-\gamma}}^{\pi} \varphi(t)\left(\frac{\sin\frac{1}{2}(n+1)t}{t}\right)^2 dt + o(1).$$

Let the above integral be denoted by I_3. Integrating it by parts, we find

$$I_3 = \left[\Phi(t)\left(\frac{\sin\frac{1}{2}(n+1)t}{t}\right)^2\right]_{n^{-\gamma}}^{\pi} - \tfrac{1}{2}(n+1)\int_{n^{-\gamma}}^{\pi} \frac{\Phi(t)}{t}\,\frac{\sin(n+1)t}{t}\,dt$$

$$+2\int_{n^{-\gamma}}^{\pi} \frac{\Phi(t)}{t}\left(\frac{\sin\frac{1}{2}(n+1)t}{t}\right)^2 dt$$

$$= O(1) - \tfrac{1}{2}(n+1)I_4 + 2I_5$$

In order to estimate the order of the integral I_5, we construct a function $\psi(t) = t^{-\eta}(\log 1/t)^{-\Delta}$, where $0<\eta<1$ being chosen previously. This function is easily seen to be monotonely decreasing in the interval $(\delta, e^{-\Delta/\eta})$ for any positive quantity δ. Writing

$$\Phi(t) = \frac{t}{(\log 1/t)^{\Delta}}\varepsilon(t).$$

where $\varepsilon(t) = o(1)$ as $t \to +0$ and

$$I_5 = \int_{n^{-\gamma}}^{e^{-\Delta/\eta}} + \int_{e^{-\Delta/\eta}}^{\pi} = I_6 + I_7,$$

we see that

$$|I_6| = \left|\int_{n^{-\gamma}}^{e^{-\Delta/\eta}} \frac{\varepsilon(t)}{(\log 1/t)^{\Delta}}\left(\frac{\sin\frac{1}{2}(n+1)t}{t}\right)^2 dt\right|$$

$$\leqq \int_{n^{-\gamma}}^{e^{-\Delta/\eta}} \frac{|\varepsilon(t)|}{t^2(\log 1/t)^{\Delta}}dt = \int_{n^{-\gamma}}^{e^{-\Delta/\eta}} \frac{\psi(t)\,|\varepsilon(t)|}{t^{2-\eta}}dt$$

$$= \psi(n^{-\gamma})\int_{n^{-\gamma}}^{\tau} \frac{|\varepsilon(t)|}{t^{2-\eta}}dt \leqq \psi(n^{-\gamma}) \max_{n^{-\gamma}<t<e^{-\Delta/\eta}} |\varepsilon(t)| \int_{n^{-\gamma}}^{\tau} t^{\eta-2}dt, \quad (n^{-\gamma}<\tau<e^{-\Delta/\eta})$$

$$= O(\psi(n^{-\gamma})n^{\gamma(1-\eta)}) = O\left(\frac{n^{\gamma}}{(\log n)^{\Delta}}\right)$$

as $n\to\infty$. $I_7 = o(1)$ by Riemann-Lebesgue's theorem. Finally, we have

$$|I_4| = \int_{n^{-\gamma}}^{\pi} \frac{\Phi}{t^2}\sin(n+1)t\,dt$$

$$= n^{2\gamma}\int_{n^{-\gamma}}^{\tau} \Phi\sin(n+1)t\,dt = O(n^{2\gamma-1}), \qquad (n^{-\gamma} < \tau < \pi)$$

$$= O((\log n)^{-\Delta})$$

since $\Phi(t)$ is of bounded variation. From the above analysis, the lemma follows.

Now, we are in a position to define a special class of functions. A function $f(x)$ is said to be of the class K if it satisfies

(a) $f(x) > 0$ $(0 \leqq x \leqq \infty)$, (b) $f(cx)/f(x)$ being uniformly bounded for $0 \leqq x < \infty$ and $\frac{1}{2} \leqq c < 2$.

The following lemma is a very useful Tauberian theorem. It may be applied to many other cases.

Lemma 2. *Let $\{u_n\}$ be a sequence of numbers. Let $\varphi(x)$ and $\psi(x)$ be two functions of the class K, If*

(i) $$u_n = o\{\varphi(n)\} \qquad (u\to\infty),$$

(ii) $$\Delta^{r+1}u_n \geqq -k\psi(n)$$

for some positive value k, where $\Delta u_n = u_{n+1} - u_n$, $\Delta^{r+1}u_n = \Delta^r u_{n+1} - \Delta^r u_n$ and

(iii) $$\varphi(n) = O\{n^2\psi(n)\}$$

then
$$\Delta^r u_n = o\{(\varphi\psi)^{\frac{1}{2}}\}$$

as $n\to\infty$.

Proof. From the conditions (i) and (ii), we may clearly suppose that

(i′) $$|u_n| < \mu(n)\varphi(n),$$

(ii′) $$\Delta^{r+1}u_n > -\lambda(n)\psi(n),$$

where $\lambda(n)$ and $\mu(n)$ are two non-increasing positive functions, $\lambda(n) = O(1)$, $\mu(n) = o(1)$ as $n\to\infty$. Taking $n > 4$, $l < \frac{1}{2}n$ and noticing that

$$\Delta^r u_n = u_{n+r} - r u_{n+r-1} + \frac{r(r-1)}{2!}u_{n+r-2} + \dots\dots + (-1)^r u_n$$

and
$$\begin{aligned}\Delta^r u_n &= \Delta^r u_{n-1} + \Delta^{r+1}u_{n-1}\\ &= \Delta^r u_{n-2} + \Delta^{r-1}u_{n-2} + \Delta^{r+1}u_{n-1}\\ &= \dots \quad \dots \quad \dots \quad \dots\\ &= \dots \quad \dots \quad \dots \quad \dots\\ &= \Delta^r u_{n-l} + \Delta^{r+1}u_{n+l} + \Delta^{r+1}u_{n-l+1} + \dots\dots + \Delta^{r+1}u_{n-1}\end{aligned}$$

we obtain

$$l\Delta^r u_n = \Delta^r u_{n-1} + \Delta^r u_{n-2} + \ldots\ldots + \Delta^r u_{n-l} + \sum_{j=1}^{l}(l-j+1)\Delta^{r+1}u_{n-j}$$

$$= A_1 u_{n+r-1} + A_2 u_{n+r-2} + \ldots\ldots + A_r u_n + A_{l+1}u_{n-l+r-1}$$

$$+ A_{l+2}u_{n-l+r-2} + \ldots\ldots + A_{l+r}u_{n-l} + \sum_{j=1}^{l}(l-j+1)\Delta^{r+1}u_{n-j}$$

where $A_1, A_2, \ldots\ldots A_r, A_{l+1}, A_{l+2}, \ldots\ldots A_{l+r}$ being a sequence of constants numerically less than a positive constant A, since the coefficient A_ν $(r < \nu < l+1)$ of $u_{n+r-\nu}$ vanishes. On the other hand, considering the relation

$$\Delta^r u_n = \Delta^r u_{n+1} - \Delta^{r+1}u_n$$

$$= \Delta^r u_{n+2} - \Delta^r u_n - \Delta^{r+1}u_{n+1}$$

$$= \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$= \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$= \Delta^r u_{n+l} - \Delta^{r+1}u_n - \Delta^{r+1}u_{n+1} - \ldots\ldots - \Delta^{r+1}u_{n+l-1}.$$

we obtain a second similar expression for $\Delta^r u_n$,

$$l\Delta^r u_n = A'_1 u_{n+1} + A'_2 u_{n+2} + \ldots\ldots + A'_r u_{n+r} + A'_{l+1}u_{n+l+1}$$

$$= + A'_{l+2}u_{n+l+2} + \ldots\ldots + A'_{l+r}u_{n+l+r} + \sum_{j=0}^{l-1}(l-j)\,\Delta^{r+1}u_{n+j},$$

where $A'_1, A'_2, \ldots\ldots, A'_r, A'_{l+1}, A'_{l+2}, \ldots\ldots A'_{l+r}$ being another sequence of constants numerically less than a positive constant A'. In views of the conditions (i′) and (ii′), it follows that

$$l\Delta^r u_n > -A\{\mu(n+r-1)\varphi(n+r-1) + \mu(n+r-2)\varphi(n+r-2) + \ldots\ldots$$

$$+ \mu(n)\varphi(n) + \mu(n-l+r-1)\varphi(n-l+r-1) + \mu(n-l+r-2)\varphi(n-l+r-2) + \ldots\ldots$$

$$+ \mu(n-l)\varphi(n-l)\} - \sum_{j=1}^{l}(l-j+1)\lambda(n-j)\psi(n-j),$$

$$l\Delta^r u_n < A'\{\mu(n+1)\varphi(n+1) + \mu(n+2)\varphi(n+2) + \ldots\ldots + \mu(n+r)\varphi(n+r)$$

$$+ \mu(n+l+1)\varphi(n+l+1) + \mu(n+l+2)\varphi(n+l+2) + \ldots\ldots$$

$$+ \mu(n+l+r)\varphi(n+l+r)\} + \sum_{j=0}^{l-1}(l-j)\lambda(n+j)\psi(n+j).$$

Considering that $\varphi(n)$ and $\psi(n)$ are both the functions of the class K and $\mu(n)$ and $\lambda(n)$ are non-increasing and positive, we find

$$l\Delta^r u_n \geq -B\mu(\tfrac{1}{2}n)\varphi(n) - A\lambda(\tfrac{1}{2}n)\psi(n)\sum_{j=1}^{l}(l-j+1)$$

$$> -B\mu(\tfrac{1}{2}n)\varphi(n) - Cl(l+1)\lambda(\tfrac{1}{2}n)\psi(n),$$

$$l\Delta^r u_n \leq B'\mu(\tfrac{1}{2}n)\varphi(n) + C'l(l+1)\lambda(\tfrac{1}{2}n)\psi(n).$$

Joining these two relations together, we obtain

$$|l\Delta^r u_n| < B''\mu(\tfrac{1}{2}n)\varphi(n) + C''l^2\lambda(\tfrac{1}{2}n)\psi(n),$$

where B'', C'' together with B', C', B and C are absolute constants independent of l and n. If we take $l = l(n) = \omega(n)\{\varphi(n)/\psi(n)\}^{\frac{1}{2}}$, where $\omega(n)$ is a positive function which makes $l(n) < \frac{1}{2}n$, then the above expression becomes

$$|\Delta^r u_n|\{\varphi(n)\psi(n)\}^{-\frac{1}{2}} < B''\mu(\tfrac{1}{2}n)(\omega(n))^{-1} + C''\lambda(\tfrac{1}{2}n)\omega(n)$$

By (iii), $\varphi(n)/\psi(n) = O(n^2)$. Since $\mu(n) = o(n)$, $\lambda(1) = O(1)$, we may choose $\omega(n)$, not greater than $\{\mu(\frac{1}{2}n)\}^{\frac{1}{2}}$ so that $l(n) < \frac{1}{2}n$ and the right side of the above expression approaches zero as n tends to infinity. Therefore

$$\Delta^r u_n = o\{(\varphi(n)\cdot\psi(n))^{\frac{1}{2}}\}.$$

The lemma is thus proved.

Proof of the theorem. Since the condition (i) of the theorem is satisfied, from Lemma 1, we have $\sigma_n = o\{(\log n)^{-\Lambda}\}$. Take, in Lemma 2, $r = 1$, $u_n = n\sigma_n$, $\varphi(n) = (n+2)(\log(n+2))^{-\Lambda}$ and $\psi(n) = (n+2)^{-1}(\log(n+2)^{\Lambda})$, then $\Delta u_n = s_n$, $\Delta^2 u_n = a_n$. It follows, from the same lemma, that $s_n = o\{(\varphi\psi)^{\frac{1}{2}}\} = o(1)$. This proves the theorem.

It seems to the author that our theorem can be improved further as follows:

Generalized theorem.* *If*

(i) $$\Phi(t) = o\{t(\log 1/t)^{-\Lambda}\} \qquad (t \to +0)$$

for some positive Λ *and*

(ii) $$na_n > -K(\log n)^{\Lambda}$$

for some positive K, *then* Σa_n *converges to* $s = o$ *and* (ii) *is the best possible condition of this kind.*

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Reference

Hardy, G. H. & Rogosinski, W. W. (1946), Fourier Series, Cambridge.

* This generalized form, provided true, includes of course, all convergence criteria of the same type. *Cf.* Hardy and Rogosinski, *loc. cit.*

ON TWO DIVERGENT DIRICHLET'S SERIES

By

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(*Communicated by the Secretary—Received February 17, 1954*)

1. In this note I consider the following divergent Dirichlet's series and show that their sum functions represent integral functions of s.

$$\sum_{2}^{\infty}\exp\left[Ain+k\frac{n}{\log n}-sn^{\beta}\right],\ 0<A<2\pi,\ 0<\beta<1 \tag{1.1}$$

$$\sum_{2}^{\infty}\exp\left[Ain^{\alpha}+k\left(\frac{n}{\log n}\right)^{\beta}-sn^{\gamma}\right],\ A>0,\ 0<\gamma<\beta\leq\alpha<1. \tag{1.2}$$

2. Let us first consider the series (1.1) and say that its sum is the limit of

$$H(s, y)=\sum_{2}^{\infty}\exp\left[Ain+(k-y)\frac{n}{\log n}-sn^{\beta}\right] \tag{2.1}$$

as $y\to 0$.

Now the series (1.1) is absolutely and uniformly convergent for $R(y)\geq k+\delta>k$ (δ being an arbitrarily small positive number) and for all bounded values of s, so that $H(s, y)$ is an integral function of s and an analytic function of y in the half-plane $R(y)\geq k+\delta>k$. What I proceed to prove is that $H(s, y)$ represents an integral function of both the variables s and y, so that $\lim_{y\to 0} H(s, y)$ is an integral function of s.

Put $y-k=y'$ and suppose, for a moment, that y' and s are both real and positive. Let z^{β} have its principal value. Also let I_1, I_2, S_1 and S_2 denote respectively the boundaries of a region S bounded by the lines $z=Re^{i\psi_1}$, $z=Re^{-i\psi_1}$ and the arcs $|\operatorname{Arg} z|\leq\psi_1$ of the circles $|z|=1+\frac{1}{2}$ and $|z|=n+\frac{1}{2}$ respectively. Then by Cauchy's theorem we may write

$$\sum_{2}^{n}\exp\left[Ain-y'\frac{n}{\log n}-sn^{\beta}\right]=\int_{s}\exp\frac{[Aiz-y'z/\log z-sz^{\beta}]}{(e^{2\pi iz}-1)}\,dz. \tag{2.2}$$

Let
$$G(s, y', z)=\frac{\exp[Aiz-y'z/\log z-sz^{\beta}]}{(e^{2\pi iz}-1)}$$

and let us proceed to show that
$$\int_{s_2}G(s, z, y')\,dz$$

converges uniformly to zero as R becomes infinite.

It is easy to see that

$$|G(s, y', z)|=O\left[\exp\left\{-AR\sin\psi-y'\frac{R\log R\cos\psi-R\psi\sin\psi}{(\log R)^2+\psi^2}\ sR^{\beta}\cos\beta\psi\right\}\right](0\leq\psi\leq\psi_1)$$

$$= O\left[\exp\left\{(2\pi - A)R\sin\psi - y'\frac{R\log R\cos\psi + R\psi\sin\psi}{(\log R)^2 + \psi^2} - sR^\beta\cos\beta\psi\right\}\right] \ (-\psi_1 \leqq \psi \leqq 0)$$

for real and positive values of s and y'.

It therefore follows that the integral along S_2 converges uniformly to zero as R becomes infinite, s and y' both remaining real and positive.

Henc for real and positive values of s and y' we may write

$$\sum_2^\infty \exp\left[Ain - y'\frac{n}{\log n} - sn^\beta\right] = \int_{I_1} + \int_{I_2} + \int_{S_1} G(s, y', z)dz \tag{2.3}$$

Now the integrand being an integral function of both the variables s and y' on the finite path S_1, the integral along S_1 represents an integral function of both s and y'.

Also for all bounded values af $s = \sigma + it$ and $y' = re^{i\theta}$, we have

$$\left|\int^\infty G(\sigma + it, re^{i\theta}, Re^{i\psi_1})dR\right| = O\left[\int^\infty \exp\left\{-AR\sin\psi_1 + r\frac{R}{\log R} + |s| R^\beta\right\}dR\right] \tag{2.4}$$

and

$$\left|\int^\infty G(\sigma + it, re^{i\theta}, Re^{-i\psi_1})dR\right| = O\left[\int^\infty \exp\left\{(A - 2\pi)R\sin\psi_1 + r\frac{R}{\log R} + |s| R^\beta\right\}dR\right] \tag{2.5}$$

Hence both the integrals along I_1 and I_2 on the right of (2.3) represent, by virtue of the conditions (2.4) and (2.5), integral functions of s and y'.

The equation (2.3) was obtained on the assumption that both s and y' are real and positive, but as the right side represents an integral function of both s and y', it gives the analytic continuation of the function represented initially by the series (2.1) in the half-plane $R(y) \geq k + \delta > k$ only. The sum-function $H(s)$ is therefore an integral function of s.

3. To consider the divergent series (1.2) we may consider the series

$$H(s, y) = \sum_2^\infty \exp\left[Ain^\alpha + (k - y)\left(\frac{n}{\log n}\right)^\beta - sn^\gamma\right],\ 0 < \gamma < \beta \leq \alpha < 1,\ A > 0. \tag{3.1}$$

and say that the sum of the series (2.2) is the limit of (3.1) as $y \to 0$.

Now the series (3.1) is absolutely and uniformly convergent for $R(y) \geq k + \delta > k$ (δ being an arbitrarily small positive number) and for all bounded values of s. so that $H(s, y)$ is an integral function of s and an analytic function of y in the half-plane $R(y) \geq k + \delta > k$. We may now proceed as in the last section and prove that $H(s, y)$ is an integral function of both s and y so that $H(s)$ is an integral function of s.

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ON THE ANALYTIC CONTINUATION OF DIRICHLET'S SERIES

By

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(*Communicated by the Secretary—Received February, 17, 1954*)

1. This note is concerned with the study of the analytic continuation of the Dirichlet series

$$\sum_{p}^{\infty} \varphi(n)\exp[-sf(n)]$$

where the functions $f(z)$ and $\varphi(z)$ are analytic in a cerain region and satisfy in that region certain conditions regarding their order of magnitude.

2. Now I prove the following theorem:

Theorem I. *If*

$$H(s) = \sum_{p}^{\infty} \varphi(n)\exp[-sf(n)]$$

where $f(n)$ and $\varphi(n)$ satisfy the following conditions

(i) $f(n)>0$ *for* $n \geq p$,

(ii) $f(z)$ and $\varphi(z)$ *are analytic functions of* $z(= x+iy = h+Re^{i\psi}$, *where* $p-1<h<p)$ *in the region* $B(h, \psi_1, \psi_2)$ *of the* z-*plane* $\psi_2 \leq Arg(z-h) \leq \psi_1$, *where* $0<\psi_1 \leq \frac{1}{2}\pi$ *and* $-\frac{1}{2} \leq \psi_2 < 0$.

(iii) *throughout the region* B *and for sufficiently large values of* $|z|$

$$f(z) \sim z/\log z$$

where $\log z$ *has its principal value and*

(iv) $$\varphi(z) = O(|e^{kf(z)}| e^{A|y|})$$

A and k being some real constants and $A<2\pi$, then

$$H(s) = G(s)+J(s)$$

where $G(s)$ is an integral function of s and

$$J(s) = \int_p^{\infty} \varphi(x)e^{-sf(x)}dx.$$

Both the Dirichlet series and the corresponding integral $\int_p \varphi(x)e^{-sf(x)}\,dx$ are absolutely and uniformly convergent in any finite part of the region $\sigma \geq k+\delta > k$. What I proceed to prove is that the difference between the functions represented by them is an integral function of s.

Suppose then, for a moment, that s is real and $\geq k+\delta > k$. With centre h and radius $R = n+\frac{1}{2}-h$, draw an arc of a circle cutting S_1 and S_2, the upper and lower parts of S at M and L respectively.

Now integrating the function

$$\{(\varphi(z)e^{-sf(z)})/(e^{2\pi iz}-1)\}$$

over the closed contour S, ie $hLNMh$, we have, by Cauchy's theorem, after some simplification,

$$\sum_{p}^{n} \varphi(n)\exp[-sf(n)] = \int_{S} \frac{\varphi(z)\exp[-sf(z)]}{(e^{2\pi iz}-1)} dz \tag{2.1}$$

$$= \int_{(hN)} F(s, z)dz + \int_{(hM)} \frac{F(s, z)}{(e^{-2\pi iz}-1)} dz + \int_{(hL)} \frac{F(s, z)}{(e^{2\pi iz}-1)} dz + \int_{(LN)} \frac{F(s, z)}{(e^{2\pi iz}-1)} dz - \int_{(NM)} \frac{F(s, z)}{(e^{-2\pi iz}-1)} dz,$$

where $F(z) = \varphi(z)\exp[-sf(z)]$.

Let us first prove that the last two integrals on the right of (2.1)$\to 0$ as $R\to\infty$, if s is supposed to be real and $>K$.

It can be easily seen that $\left|\frac{1}{e^{2\pi iz}-1}\right|$ and $\left|\frac{1}{e^{-2\pi iz}-1}\right|$ are both $<k\exp(-2\pi\,|\sin\psi|)$ on the arcs LN and NM respectively, k_1 being some constant.

Hence by virtue of the conditions (*iii*) and (*iv*), we have, over each of the two arcs S_1 and S_2, for sufficiently larges values of R, and s real and $>K$

$$|\varphi(z)[\exp-sf(z)]| = O(e^{A|y|}|\exp[(K-s)f(z)]|)$$

$$= O[\exp\{A|y|+(K-s)(p_1\overline{h+R\cos\psi}+q_1R\sin\psi)/(p_1^2+q_1^2)\}]$$

where

$$p_1 = \log\sqrt{(h^2+2hR\cos\psi+R^2)} \text{ and } q_1 = \tan^{-1}\frac{R\sin\psi}{h+R\cos\psi},$$

q_1 having its principal value.

Therefore the modulus of the integrand of each integral in question is

$$O\left[R\exp\left\{-(2\pi-A)R|\sin\psi| + \frac{(K-s)(p_1\overline{h+R\cos\psi}+q_1R\sin\psi)}{(p_1^2+q_1^2)}\right\}\right]$$

for sufficiently large values of R, and s real and $>K$.

Now, since, integration is to be performed with respect to ψ, it is easy to see that both the integrals in question $\to 0$ as $R\to\infty$, since $A<2\pi$ and $s>K$.

For s real and $>K$, we may therefore write from (2.1)

$$H(s) = \sum_{p}^{\infty} \varphi(n)e^{-sf(n)} = \int_{h}^{\infty} \varphi(x)e^{-sf(x)}dx + I_1 + I_2. \tag{2.2}$$

where
$$I_2 \equiv \int_h^{\infty(hM)} \frac{\varphi(z)e^{-sf(z)}}{(e^{-2\pi iz}-1)}\,dz \text{ and } I_1 \equiv \int_h^{\infty(hL)} \frac{\varphi(z)e^{-sf(z)}}{(e^{2\pi iz}-1)}\,dz$$

Let us now show that both I_1 and I_2 represent integral functions of s.

If s is bounded, the last integral on the right of (2.2), is
$$O\left[\exp\left\{-(2\pi-A)R\sin\psi_1+\frac{|K-s|R}{\log R}\right\}\right].$$

The integral I_2 is therefore uniformly convergent throughout any finite domain of the values of s, since $A<2\pi$. Therefore I_2 represents an integral function of s.

Similarly I_1 can be shown to represent an integral function of s. (2.2) therefore becomes
$$H(s)-\int_h^{\infty}\varphi(x)e^{-sf(x)}dx = \text{an integral function of } s. \tag{2.3}$$

But
$$\int_h^{\infty}\varphi(x)e^{-sf(x)}dx = \int_h^{p}\varphi(x)e^{-sf(x)}dx+\int_p^{\infty}\varphi(x)e^{-sf(x)}dx$$

where
$$\int_h^{p} dx \text{ is an integral function of } s.$$

We may therefore write
$$H(s)-\int_p^{\infty}\varphi(x)e^{-sf(x)}dx = G(s),$$

where $G(s)$ is an integral function of s, that is, we can write
$$H(s)-J(s) = G(s). \tag{2.4}$$

Since the right side of (2 4) is shown to be an integral function of s, it follows, from the principle of analytic continuation, that equation (2 4), obtained for s real and $>K$, persists for all values of s.

The finite singularities of $H(s)$ are therefore identical with those of the integral
$$\int_p^{\infty}\varphi(x)e^{-sf(x)}dx.$$

3. If the hypotheses of the previous theorem hold and further it is true that for sufficiently large values of $|z|$
$$f(z)\sim\left(\frac{z}{\log z}\right)^{\beta} \tag{3.1}$$

and
$$\varphi(z) = O(|e^{Kf(z)}|\,e^{-D|y|^{\alpha}}) \tag{3.2}$$

throughout S_1, where D, α and β are real positive numbers and $\beta \leq \alpha \leq 1$, then the function $H(s)$ is an integral function of s.

By Cauchy's theorem we may write for s real and $>K$

$$\sum_{p}^{n} \varphi(n) e^{-sf(n)} = \int_{hL} + \int_{LN} + \int_{NM} - \int_{hM} \frac{\varphi(z) e^{-sf(z)}}{(e^{2\pi i z}-1)}\, dz. \tag{3.3}$$

It has been shown in the previous section that $\left|\int_{LN}\right| \to 0$ as $R \to \infty$, s being real and $>K$.

We may now show that the same is true of the integral $\int_{NM}$.

Over the arc NM and the line hM, we have

$$\left|\frac{1}{e^{2\pi i z}-1}\right| < K_1 \text{ (a constant).}$$

Hence the modulus of the integrand of $\int_{(NM)}$

$$= O[R \exp\{-D|\,y\,|^{\alpha} + (l^2+\tau^2)^{\frac{1}{2}\beta}(K-s)\cos\beta\theta\}]$$

$$= O\left[R \exp\left\{-DR^{\alpha}|\sin\alpha\psi| + (K-s)\left(\frac{\varrho}{\log\varrho}\right)^{\beta}\right\}\right]$$

for sufficiently large values of R, the quantities l, τ, and θ being given by

$$l = \frac{p_1(h+R\cos\psi)+q_1R\sin\psi}{p_1^2+q_1^2}, \quad \tau = \frac{p_1R\sin\psi - q_1(h+R\cos\psi)}{p_1^2+q_1^2}$$

and $\theta = \tan^{-1}\tau/l$ and p_1 and q_1 being defined as in the previous theorem.

Therefore $$\left|\int_{(NM)}\right| \to 0 \text{ as } R \to \infty,$$

since $D>0$, $\beta \leq \alpha$ and $s>K$. Hence for s real and greater then k

$$H(s) = \sum_{p}^{\infty} \varphi(n) e^{-sf(n)} \tag{3.4}$$

$$= \int_{h}^{\infty\,(hL)} \frac{\varphi(z) e^{-sf(z)}}{(e^{2\pi i z}-1)}\, dz - \int_{h}^{\infty\,(hM)} \frac{\varphi(z) e^{-sf(z)}}{(e^{2\pi i z}-1)}\, dz.$$

The first integral on the right of (3.4) is shown in the previous section to represent an integral function of s.

Also the integrand of the last integral is

$$O\left[\exp\left\{-DR^{\alpha}\,|\sin\alpha\psi_1| + |K-s|\left(\frac{R}{\log R}\right)^{\beta}\right\}\right].$$

for sufficiently large values of R, s being bounded, so that, since $D>0$ and $\beta\leq\alpha$, the integral in question is uniformly convergent for all bounded values of s. Hence it represents an integral function of s

The function $H(s)$ represented by the series in the half-plane $\sigma\geq K+\delta>K$ is therefore continued analytically over the whole s-plane, according to the principle of analytic continuation. Our theorem is therefore established.

Examples. The following series represent integral functions of s.

(i) $$\sum_{2}^{\infty}\exp\left[Ain + (K-s)\frac{n}{\log n}\right], \qquad 0<A<2\pi$$

(ii) $$\sum_{2}^{\infty}\exp\left[Ain^{\alpha} + (K-s)\left(\frac{n}{\log n}\right)^{\beta}\right], \qquad A\neq 0,\ \beta\leq\alpha<1.$$

ON THE LINE GEOMETRY OF A CURVATURE TENSOR

By

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(*Received—May 18, 1954*)

1. **Introduction.** In his paper "On the line geometry of the Riemann tensor" Ruse (1944) used the Riemann tensor in a Riemannian V_n to define a quadratic complex of lines in an $(n-1)$-dimensional projective space. An indication of the nature of the n-dimensional theory was given in that paper and some results for a V_3 and a V_4 were obtained. In the present paper a quadratic complex of lines in an $(n-1)$-dimensional projective space has been defined by the co-variant curvature tensor corresponding to an arbitrary affine connection and results for a V_3 and V_4 have been obtained analogous to those given by Ruse in the paper referred to above.

2. **The Geometry of S_{n-1}.** Ruse considers the following projective geometry of S_{n-1}. Let V_n an n-dimensional Riemannian space with the metric

$$ds^2 = g_{ij}dx^i dx^j (i, j = 1, 2, \ldots n) \tag{2.1}$$

$P(x^i)$ be a non-singular point of V_n and let S_{n-1} be the $(n-1)$-plane at infinity in the tangent space at P. Then if X^i be a contravariant vector at P, X^i may be regarded as the homogeneous co-ordinates of a point in the projective $(n-1)$-space S_{n-1} at infinity in the centred affine tangent space at P A covariant vector a_i at P defines in S_{n-1} the $(n-2)$-plane of equation $a_iX^i = 0$. The fundamental tensor g_{ij} defines a non-degenerate $(n-2)$-quadric which will be called the fundamental quadric and has $g_{ij}X^iX^j = 0$ and $g^{ij}u_iu_j = 0$ as its point and tangential equations respectively, u_i being the current hyperplane co ordinates. If ξ^i are the co-ordinates of a point, then $\xi_i \equiv g_{ij}\xi^j$ are those of its polar $(n-2)$-plane with respect to the fundamental quadric So, in general, the raising or lowering of all suffixes of any tensor of V_n by means of the fundamental tensor corresponds in S_{n-1} to taking the polar reciprocal with respect to the fundamental quadric, of the geometrical configuration defined by that tensor. If X^i, Y^i are any two points of S_{n-1}, then $p^{ij} = X^iY^j - X^jY^i$ are the co-ordinates of the line joining them and the lines in S_{n-1} whose co-ordinates satisfy the equation

$$R_{ijkl}p^{ij}p^{kl} = 0 \tag{2.2}$$

where R_{ijkl} is the co-variant Riemann tensor, form a quadratic complex in S_{n-1}.

3. **Preliminaries.** Now, let us suppose that the V_n admits of a parallel displacement of a vecter defined by

$$dV^t + \Gamma_{ij}^t V^i dx^j = 0 \tag{3.1}$$

where Γ_{ij}^t is an arbitrary affine connection. Denote the co-variant curvature tensor corresponding to Γ_{ij}^t by F_{ikl}.

We now introduce the notions of self-associate and self-conjugate affine connections after Sen (Sen, 1950). An affine connection Γ_{ij}^t is said to be self-associate if $\Gamma_{ij}^t = \Gamma^*{}_{ij}^t$ where $\Gamma^*{}_{ij}^t = \Gamma_{ij}^t + g^{mt} g_{im,j}$, the comma followed by indices denoting the covariant derivatives of tensor with respect to Γ_{ij}^t. If, on the other hand, $\Gamma_{ij}^t = \Gamma_{ji}^t$ then Γ_{ij}^t is said to be self-conjugate. It is already known that if Γ_{ij}^t is both self-associate and self-conjugate, then $\Gamma_{ij}^t = \{{}^{\,t}_{ij}\}$ where $\{{}^{\,t}_{ij}\}$ denots the Christoffel symbol and the covariant curvature tensor F_{ijkl} formed of Γ_{ij}^t reduces to the covariant Riemann tensor R_{ijkl}. It is also known that the total number of distinct components of $R_{ijkl} = \frac{n^2(n^2-1)}{12}$. This can also be verified as follows: If Γ_{ij}^t is arbitrary then the total number of distinct components of F_{ijkl}

$$= 4\binom{n}{2} + 15\binom{n}{3} + 12\binom{n}{4} \qquad \text{where } \binom{n}{2} = {}^nC_2$$

If Γ_{ij}^t is self-associate then this number

$$= \binom{n}{2} + 6\binom{n}{3} + 6\binom{n}{4}.$$

In case Γ_{ij}^t is self-conjugate we have this number

$$= 4\binom{n}{2} + 12\binom{n}{3} + 8\binom{n}{4}.$$

Lastly, if Γ_{ij}^t is both self-associate and self-conjugate then the total number

$$= \binom{n}{2} + 3\binom{n}{3} + 2\binom{n}{4} = \frac{n^2(n^2-1)}{12}.$$

Put $$E_{ijkl} = F_{ijkl} + F_{jkil} + F_{klij} + F_{lkji} \tag{3.2}$$

Then E_{ijkl} satisfy the following identities of R_{ijkl}

$$E_{ijkl} + E_{ijlk} = 0,\ E_{ijkl} - E_{klij} = 0 \text{ and } E_{ijkl} + E_{jikl} = 0.$$

Let $${}_1F_{ij} = g^{hk} F_{hijk},\ {}_2F_{ij} = g^{hk} F_{ihkj},\ E_{ij} = g^{hk} E_{hijk}.$$

Then $$E_{ij} = g^{hk}(F_{hijk} + F_{ihkj} + F_{jkhi} + F_{kjih}) = ({}_1F_{ij} + {}_2F_{ij} + {}_2F_{ji} + {}_1F_{ji}). \tag{3.3}$$

Also put $$g_{ij} E_{ij} = E$$

Then $$g^{ij}\,{}_1F_{ij} = g^{ij} g^{hk} F_{hijk} = g^{ij} g^{hk} F_{ihkj} = g^{ij} g^{hk} F_{jkhi} = g^{ij} g^{hk} F_{kjih} = \tfrac{1}{4}E.$$

Similarly $$g^{ij}\,{}_2F_{ij} = \tfrac{1}{4}E.$$

Also, $$E^{ij} = g^{im} g^{jn} E_{mn},\ {}_1F^{ij} = g^{im} g^{jn}\,{}_1F_{mn},\ {}_2F^{ij} = g^{im} g^{jn}\,{}_2F_{mn}.$$

Therefore $$E^{ij} = g^{im} g^{jn} g^{hk} E_{hmnk} = g^{im} g^{jn} g^{hp} g^{kq} g_{pq} E_{hmnk} = g_{pq} E^{pijq}.$$

Again $${}_1E^{ij} = g^{im} g^{jn}\,{}_1E_{mn} = g^{im} g^{jn} g^{hk} F_{hmnk} = g^{im} g^{jn} g^{ph} g^{qk} g_{pq} F_{hmnk} = g_{pq} F^{pijq}.$$

Similarly $${}_2F^{ij} = g_{pq} F^{ipqj}.$$

Therefore $$E^{ij} = g_{pq} E^{pijq} = g^{pq}(F^{pijq} + F^{ipqj} + F^{jqpi} + F^{qjip}) = {}_1F^{ij} + {}_2F^{ij} + {}_2F^{ji} + {}_1F^{ji} \tag{3.4}$$

We shall use the formulas (3.3) and (3.4) hereafter. We shall also use the alternating tensor $\varepsilon_{ij\ldots kl}$, $\varepsilon^{ij\ldots kl}$ (n indices) of components $\pm\sqrt{g}$, 0 and $\pm 1/\sqrt{(g)}$, 0 respectively. They are obtainable from one another by raising and lowering suffixes in the usual way.

4. Quadratic complex $\mathbf{F}_{ijkl}$ in $\mathbf{S}_{n-1}$. If Y^i, Z^i are any two points of S_{n-1}, then $p^{ij} = Y^iZ^j - Z^iY^j$ are co-ordinates of the line joining them and the lines in S_{n-1} whose co-ordinates satisfy the equation

$$F_{ijkl}p^{ij}q^{kl} = 0. \tag{4.1}$$

constitute a quadratic complex in S_{n-1}.

Since $$E_{ijkl}p^{ij}p^{kl} = 4F_{ijkl}p^{ij}p^{kl} \qquad \text{(by 3.2)}$$

the quadratic complex (4.1) is the same as the quadratic complex

$$E_{ijkl}p^{ij}p^{kl} = 0. \tag{4.2}$$

Now, put $$L = \frac{E_{ijkl}Y^iZ^jY^kZ^l}{g_{mnpq}Y^mZ^nY^pZ^q}, \tag{4.3}$$

where $$g_{mnpq} \equiv g_{mp}g_{nq} - g_{mq}g_{np}.$$

Then, in the underlying V_n, L is the scalar at (x^i) for the orientation determined vectors Y^i, Z^i.

We have $$E_{ijkl}Y^iZ^jY^kZ^l = 2(F_{ijkl} + F_{jilk})Y^iZ^jY^kZ^l.$$

Therefore the lines of the complex

$$(F_{ijkl} + F_{jilk})Y^iZ^jY^kZ^l = 0. \tag{4.4}$$

are the projective counterpart in S_{n-1} of the orientations of the scalar $L = 0$.

Put $$L_1 = \frac{F_{ijkl}Y^iZ^jY^kZ^l}{g_{mnpq}Y^mZ^nY^pZ^q} \text{ and } L_2 = \frac{F_{jilk}Y^iZ^jY^kZ^l}{g_{mnpq}Y^mZ^nY^kZ^l}.$$

Then $$L = 2(L_1 + L_2).$$

Now, $$(E_{ijkl} - Lg_{ijkl})p^{ij}p^{kl} = 4(F_{ijkl} - \tfrac{1}{2}(L_1 + L_2)g_{ijkl})p^{ij}p^{kl}.$$

Therefore the complex $(E_{ijkl} - Lg_{ijkl})p^{ij}p^{kl} = 0$ is the same as the complex

$$[F_{ijkl} - \tfrac{1}{2}(L_1 + L_2)g_{ijkl}]p^{ij}p^{kl} = 0 \tag{4.5}$$

The lines of the complex (4.5) are therefore the projective counterpart in S_{n-1} of the orientations of the scalar $L = 2(L_1 + L_2)$.

5. The curvature tensor $\mathbf{F}_{ijkl}$ for a $\mathbf{V}_3$. (In the present section all suffixes will run from 1 to 3). In the S_2 at infinity in the tangent space at a given non-singular point $P(x^i)$ of V_3, the fundamental conic has

$$g_{ij}X^iX^j = 0,\; g^{ij}u_iu_j = 0$$

as its point and tangential equations respectively. If Y^i, Z^i are any two points of S_2, then $p^{ij} = Y^iZ^j - Z^iY^j$ are the co-ordinates of the line joining them.

Put $$p_i = \tfrac{1}{2}\varepsilon_{ijk}p^{jk} \tag{5.1}$$

then the p_i are the co-ordinates of the line in the ordinary sense, its equation in current point co-ordinates X^i being $p_i X^i = 0$. Now consider the lines p^{ij} which satisfy

$$E_{ijkl} p^{ij} p^{kl} = 0. \tag{5.2}$$

If we write $$G^{ij} = \tfrac{1}{4}\varepsilon^{imn}\varepsilon^{jpq}E_{mnpq} \tag{5.3}$$

then (5.2) can be writen as

$$G^{ij} p_i p_j = 0. \tag{5.4}$$

This shows that the lines p^{ij} in (5.2) all touch the conic of which (5.4) is the tangential equation. Since the complex

$$F_{ijkl} p^{ij} p^{kl} = 0 \tag{5.5}$$

is the same as the complex (5.2), the lines of the complex (5.5) touch the conic (5.4). This conic will be called the conic envelope F_{ijkl}

Now $$\varepsilon^{imn}\varepsilon^{jpq} = \begin{vmatrix} g^{ij} & g^{mj} & g^{nj} \\ g^{ip} & g^{mp} & g^{np} \\ g^{iq} & g^{mq} & g^{nq} \end{vmatrix}$$

Therefore it follows from (5.3) that

$$4G^{ij} = \begin{vmatrix} g^{ij} & g^{mj} & g^{nj} \\ g^{ip} & g^{mp} & g^{np} \\ g^{iq} & g^{mq} & g^{nq} \end{vmatrix} = 2\left(g^{ij}g^{mp}g^{nq}E_{mnpq} + 2g^{mj}g^{iq}g^{np}E_{mnpq}\right)$$

$$= -2g^{ij}E + 4E^{ij}$$

Therefore $$G^{ij} = E^{ij} - \tfrac{1}{2}g^{ij}E = ({}_1F^{ij} - {}_1F^{ji} + {}_2F^{ij} + {}_2F^{ji}) - 2g^{ij}F \ldots \tag{5.6}$$

where $$F = \tfrac{1}{4}E$$

The conics ${}_1\mathbf{F}_{ij}\mathbf{X}^i\mathbf{X}^j = 0$ and ${}_2\mathbf{F}_{ij}\mathbf{X}^i\mathbf{X}^j = 0$ Let Y^i be a fixed point of S_2 and let X^i be any other point on either of the tangents through it to the conic envelope determined by the lines of the complex (5.2). Then $p^{ij} = X^iY^j - X^jY^i$ satisfy the equation (5.2) whence

$$E_{ijkl} X^i Y^j X^k Y^l = 0 \tag{5.7}$$

Regarded as an equation in current point co-ordinates X^i, this represents the pair of tangents from Y^i to the conic.

Thus if $S_{ij} = E_{imnj}X^mY^n$, then $S_{ij}X^iX^j = 0$ is the equation of the pair of tangents from the pt. Y^i to the conic envelope determined by the complex (3.2). Now, consider the equation

$$E_{ij}X^iX^j = 0. \tag{5.8}$$

It represents a conic in S_2. Written in full it is

$$g^{mn}E_{imnj}X^iX^j = 0$$

From this it follows that the pair of tangents $S_{ij} = E_{imnj}X^mX^n$ from the point X^i to the conic-envelope determined by E_{ijkl} is apolar to the fundamental conic-envelope g^{ij}.

Hence the conic (5.9) or (5.8) is the locus of a point such that the tangents from it to the conic envelope E_{ijkl} and to the fundamental conic-envelope g^{ij} form a harmonic pencil.

Now, $g^{mn}E_{imnj}X^iX^j = g^{mn}(F_{imnj}+F_{mijn}+F_{njim}+F_{jnmi})X^iX^j = 2({}_1F_{ij}+{}_2F_{ij})X^iX^j$.

Hence the conic (5.9) passes through the four points of intersection of the conics

$$\left.\begin{aligned} {}_1F_{ij}X^iX^j = 0 \\ {}_2F_{ij}X^iX^j = 0 \end{aligned}\right\} \qquad (5.10)$$

The tangents from each of these four points to the conic-envelope E_{ijkl} and to the fundamental conic-envelope g^{ij} form a harmonic pencil.

But the conic-envelope E_{ijkl} is the same as the conic-envelope F_{ijkl}. We have therefore the following result:

The points of intersection of the conics (5 10) are such that the tangents from each of them to the conic-envelope F_{ijkl} and to the fundamental conic-envelope g_{ij} form a harmonic pencil.

Let us now suppose that $\Gamma_{ij}^{\ l}$ is self associate. Then $F_{ijkl}+F_{jikl} = 0$ [Sen, 1950]

Therefore ${}_1F_{ij} = {}_2F_{ij}$. Hence, in this case, each of the conics (5.10) is the locus of a point such that the tangents from it to the conic-envelope F_{ijkl} and to the fundamental conic-envelope g_{ij} form a harmonic pencil.

This result can be stated as a theorem in the following form:

If F_{ijkl} is the co-variant curvature tensor corresponding to a self-associate affine connection and $F_{ij} = g^{hk}F_{hijk}$, then the conic $F_{ij}X^iX^j = 0$ is the locus of a point such that the tangents from it to the conic-envelope F_{ijkl} and to the fundamental conic-envelope g^{ij} form a harmonic pencil.

Now, $\{{}_{i}{}^{l}{}_{j}\}$ is a self-associate affine connection, R_{ijkl} is the co-varient curvature tensor corresponding to $\{{}_{i}{}^{l}{}_{j}\}$ and $R_{ij} = g^{hk}R_{hijk}$. Hence the Ricci conic $R_{ij}X^iX^j = 0$ is the locus of a point such that the tangents from it to the conic-envelope R_{ijkl} and to the fundamental conic-envelope g^{ij} from a harmonic pencil, a result already given by Ruse in the paper mentioned earlier.

Principal directions determined by $\frac{1}{2}(\mathbf{F}_{ij}+\mathbf{F}_{ji})$. If $\Gamma_{ij}^{\ l}$ is self-associate, ${}_1F_{ij}={}_2F_{ij}=F_{ij}$ (say), then F_{ij} is not symmetric in i and j. Put $L_{ij} = \frac{1}{2}(F_{ij}+F_{ji})$. The tensor L_{ij} and the fundamental tensor g_{ij}, therefore define 4 null vectors

$$\overset{i}{\xi}_{(a)} \equiv (\overset{i}{\xi}_{(1)}\ldots\ldots\overset{i}{\xi}_{(4)})$$

at any point of V_3 each of which satisfies

$$L_{ij}\xi^i\xi^j = 0 \qquad (5.11)$$

and

$$g_{ij}\xi^i\xi^j = 0 \qquad (5.12)$$

In S_3 the ξ^i are respectively the co-ordinates of the points of intersection of (5.11) and (5.12).

Now, consider the points which have the same polar with respect to (5.11) and (5.22). If h^i be such a point

$$L_{ij}h^i = \lambda g_{ij}h^i \qquad \text{where } \lambda \text{ is a scalar.}$$

or

$$(L_{ij}-\lambda g_{ij})h^i = 0.$$

When the roots of $\det|L_{ij}-\lambda g_{ij}| = 0$ are all distinct, there will be three such points h^i.

In V_3, they determine the three principal directions corresponding to the tensor L_{ij} [Eisenhart, 1926, p. 107] We have therefore the following result:

If F_{ijkl} is the co-variant curvature tensor corresponding to a self-associate affine connection, $F_{ij} = g^{hk}F_{hijk}$, $L_{ij} = \frac{1}{2}(F_{ij}+F_{ji})$ and $\det|L_{ij}-\lambda g_{ij}| = 0$ has all the roots distinct, then there exist three points which have the same polar with respect to the conics L_{ij} and g_{ij} and they correspond in V_3 to the three principal directions determined by L_{ij} i.e., by $\frac{1}{2}(F_{ij}+F_{ji})$.

If $\Gamma_{ij}^{\ t}$ is both self-associate and self-conjugate then $\Gamma_{ij}^{\ t} = \{_{ij}^{\ t}\}$ and the following result of Ruse follows from this:

If the roots of $\det|R_{ij}-\lambda g_{ij}| = 0$ are all distinct, there exist three points which have the same polar with respect to the Ricci and fundamental conics and they correspond in V_3 to the three Ricci principal directions.

6. **Quadratic complex $\mathbf{F}_{ijkl}$ for a $\mathbf{V}_4$.** (Hereafter all suffixes run from, 1 to 4). If X^i, Y^i are two points in the S_3 at infinity associated with a non-singular point $P(x^i)$ of V_4 and if u_i, v_i are two planes passing through the line joining them, then

$$p^{ij} = X^iY^j - Y^iX^j, \quad {}^0p_{ij} = u_iv_j - v_iu_j \tag{6.1}$$

are dual sets of Plücker co-ordinates of the line.

The relation between the two sets may be written in the equivalent forms

$$p^{ij} = \tfrac{1}{2}\varepsilon^{ijkl}\,{}^0p_{kl}, \quad {}^0p_{ij} = \tfrac{1}{2}\varepsilon_{ijkl}p^{kl}. \tag{6.2}$$

In S_3, the complex determined by E_{ijkl} has dual equations

$$E_{ijkl}p^{ij}p^{kl} = 0 \qquad {}^0E^{ijkl}\,{}^0p_{ij}\,{}^0p_{kl} = 0 \tag{6.3}$$

where

$${}^0E^{ijkl} = \tfrac{1}{4}\varepsilon^{ijmn}\varepsilon^{klpq}E_{mnpq}. \tag{6.4}$$

Now,

$$E_{ijkl}p^{ij}p^{kl} = 4F_{ijkl}p^{ij}p^{kl}$$

and

$${}^0E^{ijkl}\,{}^0p_{ij}\,{}^0p_{kl} = \tfrac{1}{4}\varepsilon^{ijmn}\varepsilon^{klpq}(F_{mnpq}+F_{nmqp}+F_{pqmn}+F_{qpnm})\,{}^0p_{ij}\,{}^0p_{kl}$$

$\therefore$ in S_3, the complex determined by F_{ijkl} has dual equations

$$F_{ijkl}p^{ij}p^{kl} = 0 \qquad {}^0F^{ijkl}\,{}^0p_{ij}\,{}^0p_{kl} = 0 \tag{6.5}$$

where

$${}^0F^{ijkl} = \tfrac{1}{4}\varepsilon^{ijmn}\varepsilon^{klpq}(F_{mnpq}+F_{nmqp}+F_{pqmn}+F_{qpnm}). \tag{6.6}$$

The polar complex has dual equations

$$F^{ijkl}p_{ij}p_{kl} = 0 \text{ and } {}^0F_{ijkl}{}^0p^{ij}{}^0p^{kl} = 0 \tag{6.7}$$

The Quadrics ${}_1\mathbf{F}_{ij}\mathbf{X}^i\mathbf{X}^j = 0$ and ${}_2\mathbf{F}_{ij}\mathbf{X}^i\mathbf{X}^j = 0$. In S_3, consider the quadric

$$g^{mn}E_{imnj}X^iX^j = 0. \tag{6.8}$$

Since
$$g^{mn}E_{imnj}X^iX^j = ({}_1F_{ij} + {}_1F_{ji} + {}_2F_{ij} + {}_2F_{ji})X^iX^j$$
$$= 2({}_1F_{ij} + {}_2F_{ij})X^iX^j$$

the quadric (6.8) passes through the intersection of the quadrics

$$\left.\begin{aligned} {}_1F_{ij}X^iX^j &= 0 \\ {}_2F_{ij}X^iX^j &= 0 \end{aligned}\right\} \tag{6.9}$$

But $E_{imnj}X^iX^j$ are the co-ordinates of the complex cone of the point X^i.

So, $g^{mn}E_{imnj}X^iX^j = 0$ *i.e.*, $E_{ij}X^iX^j = 0$ is the locus of points X^i whose complex cone are outpolar to the fundamental quadric envelope g^{ij}. We have therefore the following theorem.

The points of the conic (6.9) *are such that the complex cone of each of them is outpolar to the fundamental quadric envelope g^{ij}.*

Suppose now that Γ_{ij}^t is self-associate. Then ${}_1F_{ij} = {}_2F_{ij}$. Each of the quadrics (6.9) is therefore the locus of points whose complex cones are outpolar to the fundamental quadric envelope g^{ij}.

The above result can be stated as follows:

For every self-associate affine connection of which the corresponding covariant curvature tensor is F_{ijkl} and $F_{ij} = g^{lk}F_{lnjk}$ the quadric $F_{ij}X^iX^i =$ is the locus of points whose complex cones are outpolar to the quadric envelope g^{ij}.

If, the affine connection be both self-associate and self-conjgate then $\Gamma_{ij}^t = \{{}_{i}{}^{t}{}_{j}\}$ and we get the following result of Ruse:

The Ricci quadric $R_{ij}X^iX^j = 0$ is the locus of points whose complex cones are outpolar to the quadric envelope g^{ij}.

7. **Self-polar complexes.** Let us suppose that the complex $F_{ijkl}p^{ij}p^{kl} = 0$ is self-polar with respect to the fundamental quadric. Then the complex $E_{ijkl}p^{ij}p^{kl} = 0$ is also so. Therefore the co-ordinates E^{ijkl} of the polar complex are proportional to the dual co ordinates ${}^0E^{ijkl}$ of the original complex, that is

$$KE^{ijkl} = {}^0E^{ijkl}, \quad \text{where } K \text{ is a scalar.} \tag{7.1}$$

But
$${}^0E^{ijkl} = E^{ijkl} + (g^{ik}E^{jl} + g^{jl}E^{ik} - g^{il}E^{jk} - g^{jk}E^{il}) - \tfrac{1}{2}g^{ijkl}E. \tag{7.2}$$

Therefore
$$KE^{ijkl} = E^{ijkl} + (g^{ik}E^{jl} + g^{jl}E^{ik} - g^{il}E^{jk} - g^{jk}E^{il}) - \tfrac{1}{2}g^{ijkl}$$

or
$$(K-1)E^{ijkl} = (g^{ik}E^{jl} + g^{jl}E^{ik} - g^{ik}E^{jk} - g^{jk}E^{il}) - \tfrac{1}{2}g^{ijkl}E \tag{7.3}$$

or $$(K-1)g_{jk}E^{ijkl} = g_{jk}(g^{ik}E^{jl}+\ldots\ldots\ldots\ldots)-\tfrac{1}{2}g_{jl}g^{ijkl}E$$

or $$(K-1)E^{il} = -2E^{il}+\tfrac{1}{2}g^{il}E$$

or $$(K+1)E^{il} = \tfrac{1}{2}g^{il}E = 2g^{il}F$$

Therefore $$(K+1)({}_1F^{il}+{}_2F^{il}+{}_2F^{li}+{}_1F^{li}) = 2g^{il}F \tag{7.4}$$

Hence (7.4) is the condition for the complex F_{ijkl} to be self-polar.

If $$K\neq -1, \quad (K+1)g_{il}E^{il} = 2g_{il}g^{il}F$$

or $$(K+1)E = 8F, \quad \text{or} \quad (K+1)F = 2F \quad \text{or} \quad (K-1)F = 0$$

Hence if $$F\neq 0, \ K = 1$$

whence $${}_1F^{il}+{}_2F^{il}+{}_2F^{li}+{}_1F^{li} = g^{il}F$$

or $${}_1F_{il}+{}_2F_{il}+{}_2F_{li}+{}_1F_{li}+ \ = g_{il}F \tag{7.5}$$

If $$F = 0 \text{ and } K\neq -1, \text{ then}$$

$${}_1F^{il}+{}_2F^{il}+{}_2F^{li}+{}_1F^{li} = 0 \text{ by (7.4) and (7.5) still holds.}$$

If, however $$K = -1 \quad \text{theh} \quad F = 0 \text{ by (7.4).}$$

But ${}_1F^{il}+{}_2F^{il}+{}_2F^{li}+{}_1F^{li}$ is not necessarily zero, and by the covariant form of (7.3), we get

$$E_{ijkl} = -\tfrac{1}{2}(g_{ik}E_{jl}+g_{jl}E_{ik}-g_{il}E_{jk}-g_{jk}E_{il}) \tag{7.5a}$$

or $$F_{ijkl}+F_{jilk}+F_{klij}+F_{lkji} = -\tfrac{1}{2}[g_{ik}({}_1F_{jl}+{}_2F_{jl}+{}_1F_{lj}+{}_2F_{lj})+g_{jl}({}_1F_{ik}+{}_2F_{ik}+{}_1F_{ki}+{}_2F_{ki})$$
$$-g_{il}({}_1F_{jk}+{}_2F_{jk}+{}_1F_{kj}+{}_2F_{kj})-g_{jk}({}_1F_{ij}+{}_2F_{il}+{}_1F_{li}+{}_2F_{li})] \tag{7.6}$$

Hence we have the following result.

If the complex F_{ijkl} is self-polar and such that

$$F^{ijkl}+F^{jilk}+F^{klij}+F^{lkji} = {}^0F^{ijkl} \tag{7.7}$$

then $${}_1F_{il}+{}_2F_{il}+{}_2F_{li}+{}_1F_{li}+ \ = g_{il}F \tag{7.8}$$

where the scalar F may or may not be zero.

But if the complex is self-polar and such that

$$F^{ijkl}+F^{jilk}+F^{klij}+F^{lkji} = -{}^0F^{ijkl} \tag{7.9}$$

then $$F = 0 \tag{7.10}$$

and $$F_{ijkl}+F_{jilk}+F_{klij}+F_{lkji} = -\tfrac{1}{2}[g_{ik}({}_1F_{jl}+{}_2F_{jl}+{}_1F_{lj}+{}_2F_{lj})+g_{jl}({}_1F_{ik}+{}_2F_{ik}+{}_1F_{ki}+{}_2F_{ki})$$
$$-g_{il}({}_1F_{jk}+{}_2F_{jk}+{}_1F_{kj}+{}_2F_{kj})-g_{jk}({}_1F_{il}+{}_2F_{il}+{}_1F_{li}+{}_2F_{li})] \tag{7.11}$$

Since $g_{il}FX^iX^l = 2[{}_1F_{il}+{}_2F_{il}]X^iX^l$, in S_3 (7.5) means *that the quadric $({}_1F_{il}+{}_2F_{il})X^iX^l=0$ is identical with the fundamental quadric if $F\neq 0$ or is non-existent if $F = 0$.*

If $\Gamma_{ij}{}^{t}$ is self-associate, then ${}_1F_{ij} = {}_2F_{ij} = F_{ij}$ (say) where F_{ij} is not symmetric in i and j and (7.5) reduces to

$$F_{il}+F_{li} = \tfrac{1}{2}g_{il}F$$

$$\text{i.e,}\quad \tfrac{1}{2}(F_{il}+F_{li}) = \tfrac{1}{4}g_{il}F \tag{7.12}$$

If (7.12) holds everywhere in V_4, then V_4 is homogeneous with respect to the tensor $\frac{1}{2}(F_{il}+F_{li})$[Spain, (1953), p. 25, Eisenhart, p. 114]

Further, in this case the quadric $F_{il}X^iX^l = 0$ is non existent if $F = 0$. The meaning of this is that the complex cone of every point of S_3 is outpolar to the fundamental quadric envelope g^{ij}.

If $\Gamma_{ij}{}^{t}$ is both self-associate and self-conjugate then (7.12) reduces to $R_{il} = \frac{1}{4}g_{il}R$ and if this holds every where in V_4 then V_4 is an Einstein Space. This resunt has also been given by Ruse in his paper.

Since $F = g^{ij}{}_1F_{ij} = g^{ij}{}_2F_{ij}$ the geometrical meaning of (7.10) is that either of the quadrics ${}_1F_{ij}X^iX^j = 0$, ${}_2F_{ij}X^iX^j = 0$ is outpolar to the fundamental quadric envelope g^{ij}. (7.5a) means that the complex E_{ijkl} consists of lines which meet the quadrics g_{ij} and E_{ij} in harmonically conjugate points Hence the complex F_{ijkl} consists of lines which meet quadrics g_{ij} and ${}_1F_{ij}+{}_2F_{ij}$ in harmonically conjugate points.

So, the geometrical meaning of (7.11) is that the complex F_{ijkl} consists of lines which meet the quadrics g_{ij} and ${}_1F_{ij}+{}_2F_{ij}$ in harmonically conjugate points.

8. A set of null vectors. The identities satisfied by the Plücker co-ordinates of a line can be written the form.

$$\tfrac{1}{2}\varepsilon_{ijkl}p^{ij}p^{kl} = 0 \tag{8.1}$$

Now, if the Plucker co-ordinates of a line in S_3 be interpreted in the usual way as the co-ordinates of a point on the 4-quadric (8.1) in a five-fold space then either regulus of the fundamental quadric in S_3 is represented in the five-fold space by a conic on the quadric (8.1) the plane of which will cut the quadric of equation $E_{ijkl}p^{ij}p^{kl} = 0$ in a conic. The two conics, if not coincident, will intersect in 4 points which correspond in S_3 to four lines of the regulus. Eight generators of the fundamental quadric, four of each system, therefore belong to the complex E_{ijkl} i.e to the complex F_{ijkl}. So, in general, there are 16 points on the fundamental quadric, the intersections of the two sets of 4 generators. Let ξ^i, η^i be two such points. Since both these points lie on the fundamental quadric $g_{ij}X^iX^j = 0$, they define null vectors of V_4. If they lie on the same generator of either system they are conjugate with respect to the fundamental quadric and moreover the line joining them belong to the complex E_{ijkl}.

So, $$g_{ij}\xi^i\eta^j = 0 \text{ and } E_{ijkl}\xi^i\eta^j\xi^k\eta^l = 0 \tag{8.2}$$

or $$g_{ij}\xi^i\eta^j = 0 \tag{8.3}$$

and $$(F_{ijkl}+F_{jikl})\xi^i\eta^j\xi^k\eta^l = 0 \tag{8.4}$$

On the other hand, if ξ^i, η^i do not lie on the same generator,

$$g_{ij}\xi^i\eta^j \neq 0 \text{ and } (F_{ijkl}+F_{jilk})\xi^i\eta^j\xi^k\eta^l \neq 0$$

Now, in V_4 (8.3) means that ξ^i and η^i are perpendicular.

If the four generators of the one system by numbered 1 to 4 and those of the other system be also numbered 1 to 4 and if $\xi_{(p\,q)}^{i}$ is the point of intersection of generator p of the one system with generator q of the other system then we have the following result:

In V_4, a covariant curvature tensor F_{ijkl} and the fundamental tensor g_{ij} determine in general 16 null vectors $\xi_{(p\,q)}^{i}$ such that

$$g_{ij}\xi_{(p\,q)}^{i}\xi_{(r\,s)}^{j} = 0 \text{ if either } p = r \text{ or } q = s \text{ or both} \tag{8.5}$$

and
$$(F_{ijkl}+F_{jilk})\xi_{(p\,q)}^{i}\xi_{(r\,s)}^{j}\xi_{(p\,q)}^{k}\xi_{(r\,s)}^{l} = 0 \text{ if either } p = r \tag{8.6}$$

or $q = s$ or both.

From (8.5) it follows that each of the null-vectors is perpendicular to six of the others, namely to those which have the same p or the same q as the vector in question. If Γ_{ij}^{t} self-associate then the null vectors are such that (8.5) holds and

$$F_{ijkl}\xi_{(p\,q)}^{i}\xi_{(r\,s)}^{j}\xi_{(p\,q)}^{k}\xi_{(r\,s)}^{l} = 0 \text{ if either } p = r \text{ or } q = s \text{ or both.}$$

If Γ_{ij}^{t} is both self-associate and self-conjugate we get the 16 null vectors obtained by Ruse where the vectors are such that (8.5) holds and

$$R_{ijkl}\xi_{(p\,q)}^{i}\xi_{(r\,s)}^{j}\xi_{(p\,q)}^{k}\xi_{(r\,s)}^{l} = 0 \quad \text{if either } p = r \text{ or } q = s \text{ or both.}$$

My thanks are due to Dr. R. N. Sen who has kindly directed me to the literature on the subject and taken interest in the preparation of this paper.

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References

Eisenhart, L. P. (1926), Riemannian Geometry, Princeton and Oxford.
Jessop, C. M. (1903), A Treatise on Line Complex, Cambridge.
Ruse H. S. (1944), *Proc. Royal Soc. Edin* **LXII** Part I, 64.
Sen, R. N. (1950) *Bull. Cal Math. Soc*, **42**, 177.
Sommerville, D. M. Y. (1934), Analytical Geometry of three dimensions. Cambridge.
——————, (1937), Analytical Conics, London.
Spain. B. (1953), Tensor Calculus, Edinburgh and London.

ON A MATRIX REPRESENTATION OF HOMOGENEOUS ALGEBRAIC FORMS

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(*Communicated by Prof. R. N Sen Received June 7, 1954*)

1. Several aspects of p-way matrices, their functions, factorisations and applications of their functions to polyadics, quantics etc. have been discussed by R. Oldenburger (1936, 1940), L. H. Rice (1918, 1930) and F. L. Hitchcock (1925). A p-way matrix of order n is formed of n^p elements and is written as

$$||\, a_{i_1 i_2 \cdots\cdots i_p} \,||_n \tag{1}$$

The elements $(a_{i_1 i_2 \cdots i_p})$ can be placed at points $(i_1, i_2, \ldots\ldots, i_p)$ in a linear p-space where $i_j = 1, 2, 3, \ldots\ldots, n$ for $j = 1, 2, \ldots\ldots, p$. The matrix can be separated into n-layers in any of the p-directions; a layer is a $(p-1)$-way matrix of n^{p-1} elements. According to Cayley-Rice law of multiplication (Rice, 1918, 1930), the product of a p-way matrix P of order n and a q-way matrix Q of the same order n is a $(p+q-2)$-way matrix of order n. When $p, q > 2$ the definition of multiplication depends on dividing P and Q into sets of parallel layers and sublayers till they are reduced to 2-way matrices and then multiplying the corresponding layers of the two matrices according to ordinary way of multiplication.

Using the above notions, the object of the paper is to find an application of the matrix (1) in obtaining a matrix representation of homogeneous algebraic forms of degree p in n variables. For our purpose the destinction between the cases $p = n$ and $p \neq n$ is immaterial. To simplify matters we shall consider particular cases when $p = n = 3, 4$. The general result would follow easily from the trend of argumant given in the particular cases.

2. When $p = n = 3$, we get a 3-way matrix of order 3 formed of 3^3 elements:

$$||\, a_{ijk} \,||_3 \tag{2}$$

Let us suppose that the indices i, j, k correspond respectively to directions which give rows, columns and normals Then the matrix (2) can be divided into 3 row-column layers in which the third index remains fixed and into 3 row-normal layers in which the second index remains fiixed and into 3 column-normal layers in which the first index remains fixed. Rows, columns and normals are generally called files. These three sets of parallel layers running respectively in three directions are 2-way matrices of elements and the three sets of parallel files are 1-way matrices of elements. Two layers in different directions intersect in a file and three layers in different directions intersect in a single element.

Let us now suppose that the elements a_{ijk} of (2) belong to a field F and satisfy the condition that they are symmetric in the indices i, j, k. This means that the matrix (2)

is symmetric in the diagonal cantaining the elements $a_{111}, a_{222}, a_{333}$. It therefore follows that (2) can be represented by the three ordered parallel layers corresponding to any one of the three directions. Thus the symmetric matrix (2) can be represented without ambiguity by

$$S = \left\| \begin{array}{ccc|ccc|ccc} a_{111} & a_{112} & a_{113} & a_{211} & a_{212} & a_{213} & a_{311} & a_{312} & a_{313} \\ a_{121} & a_{122} & a_{123} & a_{221} & a_{222} & a_{223} & a_{321} & a_{322} & a_{323} \\ a_{131} & a_{132} & a_{133} & a_{231} & a_{232} & a_{233} & a_{331} & a_{332} & a_{333} \end{array} \right\| \tag{3}$$

Now using Cayley-Rice law of multiplication, let us multiply the 3-way symmetric matrix S given by (3) from the left by the 1-way matrix $X = \| x_1, x_2, x_3 \|$ of order 3. If A_1, A_2, A_3 are the layers of S as given in (3), we obtain the following 2-way matrix of order 3:

$$XS = \| XA_1 | XA_2 | XA_3 \|$$

$$= \| \Sigma x_j a_{1j1}, \Sigma x_j a_{1j2}, \Sigma x_j a_{1j3} | \Sigma x_j a_{2j1}, \Sigma x_j a_{2j2}, \Sigma x_j a_{2j3} | \Sigma x_j a_{3j1}, \Sigma x_j a_{3j2}, \Sigma x_j a_{3j3} \| \tag{4}$$

or, putting $b_{ik} = \Sigma_j x_j a_{ijk}$, (4) can be written as

$$\| b_{11}, b_{12}, b_{13} | b_{21}, b_{22}, b_{23} | b_{31}, b_{32}, b_{33} \| \tag{4'}$$

where $b_{ik} = b_{ki}$. Just as the matrix (2) was represented by the form (3) and as this representation is a one-to-one, so the form (4′) *i.e.* (4) can be rearranged to form the following ordinary 2-way symmetric matrix,

$$XS = \left\| \begin{array}{ccc} \Sigma x_j a_{1j1} & \Sigma x_j a_{1j2} & \Sigma x_j a_{1j3} \\ \Sigma x_j a_{2j1} & \Sigma x_j a_{2j2} & \Sigma x_j a_{2j3} \\ \Sigma x_j a_{3j1} & \Sigma x_j a_{3j2} & \Sigma x_j a_{3j3} \end{array} \right\| \tag{5}$$

It now immediately follows that if X^t is the transposed of X, then,

$$X(XS)X^t = \Sigma a_{ijk} x_i x_j x_k.$$

It can be verified that the same cubic form can be expressed by

$$X(SX^t)X^t.$$

From the nature of multiplication as used above it follows that a homogeneous cubic form $\Sigma a_{ijk} x_i x_j x_k$ in 3 variables can be represented in the matix form

$$X(XS)X^t = X(SX^t)X^t, \tag{6}$$

where S is a 3-way *symmetric* matrix of order 3.

3. We shall now consider the case when $p = n = 4$ Remarks on the 4-way matrix of order 4 similar to those on the 3-way matrix of order 3 can easily be made with, of course, corresponding change of notations. Let us suppose that S is a 4-way matrix of order 4 whose elements a_{ijkl} are symmetric in all the four indices. This implies that every layer and sublayer of S is a symmetric matrix. Then as in the last article we can write

$$S = \| P_1 | P_2 | P_3 | P_4 \|$$

where the parallel layers P_i, $i = 1, 2, 3, 4,$ are 3-way symmetric matrices of order 4.

Now each of layers P_i, $i = 1, 2, 3, 4$, can again be subdivided into 4 ordered parallel layers $P_{i1}, P_{i2}, P_{i3}, P_{i4}$ which are 2-way symmetric matrices of order 4 and therefore can be represented uniquely by

$$\|P_{i1} \mid P_{i2} \mid P_{i3} \mid P_{i4}\|, \ i = 1, 2, 3, 4.$$

If $X = \| x_1, x_2, x_3, x_4 \|$ is a 1-way matrix of order 4, then

$$XP_i = \| XP_{i1} \mid XP_{i2} \mid XP_{i3} \mid XP_{i4} \|$$

Now according to our definition of multiplication XP_i is a 2-way symmetric matrix of order 4 which can be represented by

$$\| XP_i \| = \left\| \begin{matrix} \Sigma x_k a_{i_1 k_1} & \Sigma x_k a_{i_1 k_2} & \Sigma x_k a_{i_1 k_3} & \Sigma x_k a_{i_1 k_4} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \Sigma x_k a_{i_4 k_1} & \Sigma x_k a_{i_4 k_2} & \Sigma x_k a_{i_4 k_3} & \Sigma x_k a_{i_4 k_4} \end{matrix} \right\|$$

$$= \left\| \begin{matrix} b_{i11} & b_{i12} & b_{i13} & b_{i14} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ b_{i41} & b_{i42} & b_{i43} & b_{i44} \end{matrix} \right\| = B_i(\text{say}), \ i = 1, 2, 3, 4.$$

Therefore, $$XS = \| B_1 \mid B_2 \mid B_3 \mid B_4 \|,$$

which is a 3-way symmetric matrix of order 4. Further multiplying by X from the left

$$X(XS) = \| XB_1 \mid XB_2 \mid XB_3 \mid XB_4 \|.$$

As before XB_i, $i = 1, 2, 3, 4$, is a 1-way matrix of order 4:

$$\| \Sigma x_j b_{ij1} \quad \Sigma x_j b_{ij2} \quad \Sigma x_j b_{ij3} \quad \Sigma x_j b_{ij4} \|$$

$$= \| c_{i1} \quad c_{i2} \quad c_{i3} \quad c_{i4} \| = c_i(\text{say}), \ i = 1, 2, 3, 4.$$

Thus $X(XS) = \| C_1 \mid C_2 \mid C_3 \mid C_4 \|$ is a 2-way symmetric matrix of order 4, which can be represented as:

$$\left\| \begin{matrix} c_{11} & c_{12} & c_{13} & c_{14} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ c_{41} & c_{42} & c_{43} & c_{44} \end{matrix} \right\|$$

It is now immediately seen that if X^t is the transposed of X, then

$$X(X(XS))X^t = \Sigma a_{ijkl} x_i x_j x_k x_l \tag{7}$$

It can also be verified that the same biquadratic can be expressed by

$$X((XS)X^t)X^t = X((SX^t)X^t)X^t = X(X(SX^t))X^t$$

Thus it follows that a homogeneous biquadratic form $\Sigma a_{ijkl}x_ix_jx_kx_l$ in four variables can be expressed be

$$X(X(XS))X^t = X((XS)X^t)X^t = X(X(SX^t))X^t = X((SX^t)X^t)X^t \tag{8}$$

where S is a 4-way *symmetric* matrix of order 4.

Following the same trend of argument it follows that the matrix

$$X(X......(X(XS))......)X^t = X(......((SX^t)X^t)......X^t)X^t \tag{9}$$

represents a homogeneous algebraic form of degree p in n variables where S is a p-way *symmetric* matrix of order n, $X = ||x_1, x_2, x_3...x_n||$ is a 1-way matrix of order n and X^t its transposed. In the first representation of (9) there are $p-1$ factors X to the left of S and one X^t to the right and in the second representation there are $p-1$ factors X^t to the right of S and one X to the left.

In conclusion, I beg to tender my grateful thanks to Dr. R. N. Sen for his suggestions, help and guidance in the preparation of the paper.

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References

Hitchcock, F. L. (1925), *J. for Math. & Phys. Mass. I. Tech.*, **IV**, 238.
Oldenburger, R , (1936), *Trans. Am Math. Soc.*, **39**, 422.
——————, (1940), *Am. Math. Month* , **47**, 25
Rice, L. H , (1918), *Am. J. Math.*, **XL**, 212.
——————, (1920), *J. for Math. & Phys Mass. I. Tech* , **IX**, 47.

SOME PROBLEMS OF PLANE STRAIN IN A CYLINDRICALLY AEOLOTROPIC CYLINDER

By

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(*Communicated by Dr. S. Ghosh—Received June 7, 1954*)

In a recent paper Carrier (1943) has developed the approximate theory of the stretching of a cylindrically aeolotropic plate by forces acting in the plane of the plate. As an exact determination of the stress system satisfying the stress equations of equilibrium and the compatibility conditions is very difficult, Carrier has determined only the average stresses in a plate on the assumption that three of the components of average stress vanish. The difficulty encountered by Carrier does not arise in problems of plane strain where we start with the displacement instead of with the stresses, so that the compatibility conditions are automatically satisfied. In the present paper the following problems of plane strain in a cylindrically aeolotropic material in the most general sense are considered. (1) the rotation of a hollow circular cylinder about its axis, (2) the stretching of a hollow circular cylinder in the direction of its axis, and (3) the intial stresses in a hollow circular cylinder with radial fissure.

1. Introducing cylindrical coordinates r, θ, z we have for the strain energy function the expression

$$W = \tfrac{1}{2}c_{11}e_{rr}^2 + \tfrac{1}{2}c_{22}e_{\theta\theta}^2 + \tfrac{1}{2}c_{33}e_{zz}^2 + c_{23}e_{\theta\theta}e_{zz} + c_{31}e_{zz}e_{rr} + c_{12}e_{rr}e_{\theta\theta} + \tfrac{1}{2}c_{44}e_{\theta z}^2 + \tfrac{1}{2}c_{55}e_{zr}^2 + \tfrac{1}{2}c_{66}e_{r\theta}^2 \tag{1}$$

The stress components are then given by

$$\left.\begin{aligned}
\widehat{rr} &= c_{11}e_{rr} + c_{12}e_{\theta\theta} + c_{13}e_{zz}\\
\widehat{\theta\theta} &= c_{21}e_{rr} + c_{22}e_{\theta\theta} + c_{23}e_{zz}\\
\widehat{zz} &= c_{31}e_{rr} + c_{32}e_{\theta\theta} + c_{33}e_{zz}\\
\widehat{\theta z} &= c_{44}e_{\theta z},\ \widehat{zr} = c_{55}e_{zr},\ \widehat{r\theta} = c_{66}e_{r\theta}
\end{aligned}\right\} \tag{2}$$

where $\quad c_{ij} = c_{ji}$.

In strains symmetrical about the z-axis, we have only the radial displacement $u_r = u$ and the longitudinal displacement $u_z = w$ and these are independent of θ. In plane strain, in planes perpendicular to the z-axis, u is a function of r and $w = ez$, where e is the extension (constant) in the direction of the z-axis. Then we have from the expressions of the strain components in cylindrical co-ordinates (Love, 1944, p 56)

$$\left.\begin{aligned}
e_{rr} &= \frac{\partial u}{\partial r},\ e_{\theta\theta} = \frac{u}{r},\quad e_{zz} = e\\
e_{\theta z} &= e_{zr} = e_{r\theta} = 0.
\end{aligned}\right\} \tag{3}$$

Substituting in (2) we get

$$\left.\begin{aligned}\widehat{rr} &= c_{11}\frac{\partial u}{\partial r}+c_{12}\frac{u}{r}+c_{13}e\\ \widehat{\theta\theta} &= c_{21}\frac{\partial u}{\partial r}+c_{22}\frac{u}{r}+c_{23}e\\ \widehat{zz} &= c_{31}\frac{\partial u}{\partial r}+c_{32}\frac{u}{r}+c_{33}e\\ \widehat{\theta z} &= \widehat{zr} = \widehat{r\theta} = 0.\end{aligned}\right\} \tag{4}$$

When the cylinder rotates about the z-axis, with angular velocity ω the stress equations in cylindrical co-ordinates are (Love, 1944, p. 90)

$$\left.\begin{aligned}&\frac{\partial\widehat{rr}}{\partial r}+\frac{1}{r}\frac{\partial\widehat{r\theta}}{\partial\theta}+\frac{\partial\widehat{rz}}{\partial r}+\frac{\widehat{rr}-\widehat{\theta\theta}}{r}+\omega^2\varrho r = 0\\ &\frac{\partial\widehat{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\widehat{\theta\theta}}{\partial\theta}+\frac{\partial\widehat{\theta z}}{\partial z}+\frac{2\widehat{r\theta}}{r} = 0\\ &\frac{\partial\widehat{rz}}{\partial r}+\frac{1}{r}\frac{\partial\widehat{\theta z}}{}+\frac{\partial\widehat{zz}}{\partial z}+\frac{\widehat{rz}}{r} = 0\end{aligned}\right\} \tag{5}$$

Substituting from (4), we see that the last two equations of (5) are identically satisfied and the first equation reduces to

$$c_{11}\frac{\partial^2 u}{\partial r^2}+\frac{c_{11}}{r}\frac{\partial u}{\partial r}-\frac{c_{11}}{r^2}u+(c_{13}-c_{23})\frac{e}{r}+\omega^2\varrho r = 0 \tag{6}$$

This equation gives

$$u = Ar^{\alpha}+Br^{-\alpha}+aer+br^3 \tag{7}$$

where $\qquad \alpha = \sqrt{c_{22}/c_{11}}, \quad a = (c_{23}-c_{13})/(c_{11}-c_{22}), \quad b = -\omega^2\varrho/(9c_{11}-c_{22}) \qquad (8)$

and A and B are two constants.

We have from (4) and (7)

$$\left.\begin{aligned}\widehat{rr} &= a_1Ar^{\alpha-1}+b_1Br^{-\alpha-1}+c_1e+d_1r^2\\ \widehat{\theta\theta} &= a_2Ar^{\alpha-1}+b_2Br^{-\alpha-1}+c_2e+d_2r^2\\ \widehat{zz} &= a_3Ar^{\alpha-1}+b_3Br^{-\alpha-1}+c_3e+d_3r^2\end{aligned}\right\} \tag{9}$$

where

$$\left.\begin{aligned}&a_i = (c_{i1}\alpha+c_{i2}), \qquad b_i = (-c_{i1}\alpha+c_{i2})\\ &c_i = (c_{i1}+c_{i2})a+c_{i3}, \quad d_i = (3c_{i1}+c_{i2})b\\ &i = 1, 2, 3.\end{aligned}\right\} \tag{10}$$

Assuming that $\widehat{rr} = 0$ on the boundaries $r = r_1$ and $r = r_2$ of the cylinder we have,

$$a_1 A r_1^{\alpha-1} + b_1 B r_1^{-\alpha-1} + c_1 e + d_1 r_1^2 = 0$$

$$a_1 A r_2^{\alpha-1} + b_1 B r_2^{-\alpha-1} + c_1 e + d_1 r_2^2 = 0$$

from which we get

$$\left.\begin{aligned} a_1 A &= k c_1 e(r_1^{-\alpha-1} - r_2^{-\alpha-1}) + k d_1 (r_1^{-\alpha-1} r_2^2 - r_2^{-\alpha-1} r_1^2) \\ b_1 B &= -k c_1 e(r_1^{\alpha-1} - r_2^{\alpha-1}) - k d_1 (r_1^{\alpha-1} r_2^2 - r_2^{\alpha-1} r_1^2) \end{aligned}\right\} \tag{11}$$

where

$$\frac{1}{k} = r_1^{\alpha-1} r_2^{-\alpha-1} - r_2^{\alpha-1} r_1^{-\alpha-1} \tag{12}$$

As in the isotropic case the tractions $\widehat{zz}$ at the ends of the cylinder cannot be made to vanish, but can be so adjusted that they have no statical resultant, *i.e.*

$$\int_{r_1}^{r_2} \widehat{zz}\, r dr = 0$$

Substituting from (9) we get

$$\frac{a_3 A(r_2^{\alpha+1} - r_1^{\alpha+1})}{\alpha+1} + \frac{b_3 B(r_2^{-\alpha+1} - r_1^{-\alpha+1})}{-\alpha+1} + \tfrac{1}{2} c_3 e(r_2^2 - r_1^2) + \tfrac{1}{4} d_3 (r_2^4 - r_1^4) = 0 \tag{13}$$

Substituting for A and B from (11), we get the value of e. When e is known A and B are determined from (11).

2. If we put $\omega = 0$ and therefore $d_1 = d_2 = d_3 = 0$, we get the longitudinal stretching of the hollow cylinder. The problem of longitudinal stretching for an ordinary cylindrically aeolotropic cylinder has been given by Lekhnitsky (1948). We get, instead of (11), the following equations

$$\left.\begin{aligned} a_1 A &= k c_1 e(r_1^{-\alpha-1} - r_2^{-\alpha-1}) \\ b_1 B &= -k c_1 e(r_1^{\alpha-1} - r_2^{\alpha-1}) \end{aligned}\right\} \tag{14}$$

If T be the given longitudinal tension,

$$\int_{r_1}^{r_2} \widehat{zz}\, r dr = T$$

or

$$\frac{a_3 A(r_2^{\alpha+1} - r_1^{\alpha+1})}{\alpha+1} + \frac{b_3 B(r_2^{-\alpha+1} - r_1^{-\alpha+1})}{-\alpha+1} + \tfrac{1}{2} c_3 e(r_2^2 - r_1^2) = T \tag{15}$$

The equations (14) and (15) determine A, B and e.

3. If we take

$$u = f(r), \quad v = Er\theta \text{ and } w = ez \tag{16}$$

we see that the displacement is discontinuous across a radial line, the discontinuity being $2\pi Er$ normal to this line. Such a displacement arises in a hollow cylinder when a thin slice of the material bounded by the lines $y = \pm \pi Ex$ is removed and the plane faces are subsequently joined.

The stresses calculated from this displacement are

$$\left.\begin{aligned}\widehat{rr} &= c_{11}\frac{\partial u}{\partial r}+c_{12}\left(\frac{u}{r}+E\right)+c_{13}e\\ \widehat{\theta\theta} &= c_{21}\frac{\partial u}{\partial r}+c_{22}\left(\frac{u}{r}+E\right)+c_{23}e\\ \widehat{zz} &= c_{31}\frac{\partial u}{\partial r}+c_{32}\left(\frac{u}{r}+E\right)+c_{33}e\\ \widehat{\theta z} &= \widehat{rz} = \widehat{r\theta} = 0\end{aligned}\right\} \tag{17}$$

The first of the stress equations of equilibrium reduces to

$$c_{11}\frac{\partial^2 u}{\partial r^2}+\frac{c_{11}}{r}\frac{\partial u}{\partial r}-c_{22}\frac{u}{r^2}+(c_{12}-c_{22})\frac{E}{r}+(c_{13}-c_{23})\frac{e}{r}=0 \tag{18}$$

while the other two are identically satisfied.

A solution of this equation is

$$u = Ar^{\alpha}+Br^{-\alpha}+acr+cEr \tag{19}$$

where α and a are given by (8) and

$$c = (c_{22}-c_{12})/(c_{11}-c_{22}).$$

Then

$$\left.\begin{aligned}\widehat{rr} &= a_1Ar^{\alpha-1}+b_1Br^{-\alpha-1}+c_1e+e_1E\\ \widehat{\theta\theta} &= a_2Ar^{\alpha-1}+b_2Br^{-\alpha-1}+c_2e+e_2E\\ \widehat{zz} &= a_3Ar^{\alpha-1}+b_3Br^{-\alpha-1}+c_3e+e_3E\end{aligned}\right\} \tag{20}$$

where a_i, b_i, c_i are given by (10) and

$$e_i = (c_{i1}+c_{i2})c+c_{i2}.$$

The condition that the surfaces $r = r_1$ and $r = r_2$ are free from stress gives

$$\left.\begin{aligned}a_1A &= k(c_1e+e_1E)(r_1^{-\alpha-1}-r_2^{-\alpha-1})\\ b_1B &= -k(c_1e+e_1E)(r_1^{\alpha-1}-r_2^{\alpha-1})\end{aligned}\right\} \tag{21}$$

where k is given by (12).

The condition that the tractions across a plane z = constant have a zero resultant is

$$\frac{a_3A(r_2^{\alpha+1}-r_1^{\alpha+1})}{\alpha+1}+\frac{b_3B(r_2^{-\alpha+1}-r_1^{-\alpha+1})}{-\alpha+1}+\tfrac{1}{2}(c_3e+e_3E)(r_2^2-r_1^2)=0 \tag{22}$$

The equations (21) and (22) determine A, B and e.

In conclusion I express my gratefulness to Dr. S. Ghosh of University College of Science for his guidance in preparing this paper.

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References

Carrier, G. F. (1943), *Trans. A. S. M. E.*, **65**, *A*, 117.
Lekhnitsky, S. G. (1948), *J. Appl. Math. Mech.* **13**, 307.
Love, A. E. H. (1944), *A Treatise on the Mathematical Theory of Elasticity*, 4th. ed. Dover Publications.

VIBRATIONS OF SPHERICALLY AEOLOTROPIC SHELL

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(*Communicated by Dr. S. Ghosh— Received July 16, 1954*)

Very few problems of spherically aeolotropic elastic material have been considered so far because of the inherent difficulty in solving complicated simultaneous partial differential equations. Saint Venant (1865) has considered the equilibrium of a spherically aeolotropic shell under internal and external pressure and Seth (1946) has extended the solution to the case of finite strain In the present paper two problems of vibration of spherical shell of spherically aeolotropic material have been considered, *viz.* (1) radial vibration of the shell and (2) rotatory vibration of the shell. In the solution of both these problems as well as those of Saint-Venant and Seth the success is due to the fact that only one of the components of displacement is non-zero, so that two of the equations of motion or equilibrium are identically satisfied whereas the third reduces to an ordinary differential equation of the second order.

Introducing polar co-ordinates r, θ, φ the strain energy function for a spherically aeolotropic material can be written as (Love, 1944, p. 160)

$$W = \tfrac{1}{2}c_{11}e^2_{rr} + \tfrac{1}{2}c_{22}(e^2_{\theta\theta} + e^2_{\varphi\varphi}) + c_{12}(e_{\theta\theta} + e_{\varphi\varphi})e_{rr}$$
$$+ c_{23}e_{\theta\theta}e_{\varphi\varphi} + \tfrac{1}{2}c_{44}e^2_{\theta\varphi} + \tfrac{1}{2}c_{55}(e^2_{\varphi r} + e^2_{r\theta}) \qquad (1)$$

where $$c_{23} = c_{22} - 2c_{44}.$$

The stresses are therefore given by

$$\left.\begin{aligned}
\widehat{rr} &= c_{11}e_{rr} + c_{12}(e_{\theta\theta} + e_{\varphi\varphi}) \\
\widehat{\theta\theta} &= c_{12}e_{rr} + c_{22}e_{\theta\theta} + c_{23}e_{\varphi\varphi} \\
\widehat{\varphi\varphi} &= c_{12}e_{rr} + c_{23}e_{\theta\theta} + c_{22}e_{\varphi\varphi} \\
\widehat{\theta\varphi} &= c_{44}e_{\theta\varphi},\ \widehat{\varphi r} = c_{55}e_{\varphi r},\ \widehat{r\theta} = c_{55}e_{r\theta}
\end{aligned}\right\} \qquad (2)$$

The components of strain in polar co-ordinates are given by (Love, 1944, p. 56)

$$\left.\begin{aligned}
e_{rr} &= \frac{\partial u_r}{\partial r},\ e_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r},\ e_{\varphi\varphi} = \frac{1}{r\sin\theta}\frac{\partial u_\varphi}{\partial \varphi} + \frac{u_\theta}{r}\cot\theta + \frac{u_r}{r} \\
e_{\theta\varphi} &= \frac{1}{r}\left(\frac{\partial u_\varphi}{\partial\theta} - u_\varphi\cot\theta\right) + \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\varphi} \\
e_{\varphi r} &= \frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\varphi} + \frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r},\ e_{r\theta} = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial\theta}.
\end{aligned}\right\} \qquad (3)$$

The stress equations of motion are (Love, 1944, p. 91)

$$\left.\begin{aligned}
&\frac{\partial\widehat{rr}}{\partial r}+\frac{1}{r}\frac{\partial\widehat{r\theta}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial\widehat{r\varphi}}{\partial\varphi}+\frac{1}{r}(2\widehat{rr}-\widehat{\theta\theta}-\widehat{\varphi\varphi}+\widehat{r\theta}\cot\theta)=\varrho\ddot{u}_r\\
&\frac{\partial\widehat{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\widehat{\theta\theta}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial\widehat{\theta\theta}}{\partial\varphi}+\frac{1}{r}\{(\widehat{\theta\theta}-\widehat{\varphi\varphi})\cot\theta+3\widehat{r\theta}\}=\varrho\ddot{u}_\theta\\
&\frac{\partial\widehat{r\varphi}}{\partial r}+\frac{1}{r}\frac{\partial\widehat{\theta\varphi}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial\widehat{\varphi\varphi}}{\partial\varphi}+\frac{1}{r}\{3\widehat{r\varphi}+2\widehat{\theta\varphi}\cot\theta\}=\varrho\ddot{u}_\varphi
\end{aligned}\right\} \tag{4}$$

1. For the radial vibration of the spherical shell we assume

$$u_r = Ue^{ipt}, \quad u_\theta = 0, \quad u_\varphi = 0 \tag{5}$$

where U is a function of r only.

The strain components are then

$$e_{rr} = \frac{\partial U}{\partial r}e^{ipt}, \quad e_{\theta\theta} = \frac{U}{r}e^{ipt}, \quad e_{\varphi\varphi} = \frac{U}{r}e^{ipt} \tag{6}$$

$$e_{\theta\varphi} = e_{\varphi r} = e_{r\theta} = 0$$

The stress components are

$$\widehat{rr} = \left(c_{11}\frac{\partial U}{\partial r}+2c_{12}\frac{U}{r}\right)e^{ipt}$$

$$\widehat{\theta\theta} = \widehat{\varphi\varphi} = \left[c_{12}\frac{\partial U}{\partial r}+(c_{22}+c_{23})\frac{U}{r}\right]e^{ipt} \tag{7}$$

$$\widehat{\theta\varphi} = \widehat{\varphi r} = \widehat{r\theta} = 0$$

Substituting in equations (4) we see that the second and third equations are identically satisfied and the first becomes

$$\frac{\partial^2 U}{\partial r^2}+\frac{2}{r}\frac{\partial U}{\partial r}+\frac{2}{c_{11}}(c_{12}-c_{22}-c_{23})\frac{U}{r^2}+\frac{\varrho p^2}{c_{11}}U = 0. \tag{8}$$

Putting $U = r^{-\frac{1}{2}}V$ we get

$$\frac{\partial^2 V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial r}+\left(k^2-\frac{n^2}{r^2}\right)V = 0 \tag{9}$$

where $$n^2 = \tfrac{1}{4}\left[1+\frac{8}{c_{11}}(c_{22}+c_{23}-c_{11})\right] \text{ and } k^2 = \frac{\varrho p^2}{c_{11}} \tag{10}$$

A solution of (9) is

$$V = AJ_n(kr)+BY_n(kr)$$

where J_n and Y_n are Bessel's functions of the first and second kind respectively of order n. This gives

$$U = r^{-\frac{1}{2}}[AJ_n(kr)+BY_n(kr)] \tag{11}$$

Calculating $\widehat{rr}$ from (7) and (11) we get

$$\widehat{rr} = r^{-3/2}[c_{11}kr\{AJ'_n(kr)+BY'_n(kr)\}+\tfrac{1}{2}(4c_{12}-c_{11})\{AJ_n(kr)+BY_n(kr)\}]e^{ipt}.$$

Using the recurrence formulae

$$J'_n(z) = J_{n-1}(z)-(n/z)J_n(z)$$

$$Y'_n(z) = Y_{n-1}(z)-(n/z)J_n(z)$$

we get

$$\widehat{rr} = r^{-3/2}[A\{c_{11}krJ_{n-1}(kr)+\tfrac{1}{2}(4c_{12}-c_{11}-2nc_{11})J_n(kr)\} + B\{c_{11}krY_{n-1}(kr)+\tfrac{1}{2}(4c_{12}-c_{11}-2nc_{11})Y_n(kr)\}]e^{ipt}. \tag{12}$$

If the boundaries of the shell be $r = a$ and $r = b$, we have $\widehat{rr} = 0$ when $r = a$ and $r = b$.

$$A\{2c_{11}kaJ_{n-1}(ka)+(4c_{12}-c_{11}-2nc_{11})J_n(ka)\} + B\{2c_{11}kaY_{n-1}(ka)+(4c_{12}-c_{11}-2nc_{11})Y_n(ka)\} = 0 \tag{13}$$

$$A\{2c_{11}kbJ_{n-1}(kb)+(4c_{12}-c_{11}-2nc_{11})J_n(kb)\} + B\{2c_{11}kbY_{n-1}(kb)+(4c_{12}-c_{11}-2nc_{11})Y_n(kb)\} = 0 \tag{14}$$

Eliminating A and B from (13) and (14) we get the frequency equation as

$$\frac{2c_{11}kaJ_{n-1}(ka)+(4c_{12}-c_{11}-2nc_{11})J_n(ka)}{2c_{11}kaY_{n-1}(ka)+(4c_{12}-c_{11}-2nc_{11})Y_n(ka)} = \frac{2c_{11}kbJ_{n-1}(kb)+(4c_{12}-c_{11}-2nc_{11})J_n(kb)}{2c_{11}kbY_{n-1}(kb)+(4c_{12}-c_{11}-2nc_{11})Y_n(kb)} \tag{15}$$

2. For rotatory vibration of the shell we assume

$$u_r = 0, \quad u_\theta = 0, \quad u_\varphi = f(r)\sin\theta e^{ipt}. \tag{16}$$

We then have

$$e_{rr} = e_{\theta\theta} = e_{\varphi\varphi} = e_{\theta\varphi} = e_{r\theta} = 0$$

$$e_{\varphi r} = \left[f'(r)-\frac{f(r)}{r}\right]\sin\theta\, e^{ipt} \tag{17}$$

so that

$$\widehat{rr} = \widehat{\theta\theta} = \widehat{\varphi\varphi} = \widehat{\theta\varphi} = \widehat{r\theta} = 0$$

$$\widehat{\varphi r} = c_{55}\left[f'(r)-\frac{f(r)}{r}\right]\sin\theta\, e^{ipt} \tag{18}$$

Substituting in (4) we see that the first and second equations are identically satisfied while the third reduces to

$$\frac{\partial^2 f}{\partial r^2}+\frac{2}{r}\frac{\partial f}{\partial r}-\frac{2}{r^2}f+\frac{\varrho p^2}{c_{55}}f = 0$$

Proceeding exactly as in section 1 we get

$$f(r) = r^{-\frac{1}{2}}[AJ_n(kr)+BJ_{-n}(kr)] \tag{19}$$

where

$$n = 3/2 \text{ and } k^2 = \frac{\varrho p^2}{c_{55}}. \tag{20}$$

We have then from (18)

$$\widehat{\varphi r} = c_{55} r^{-3/2}[kr\{AJ'_n(kr) + BJ'_{-n}(kr)\} - (3/2)\{AJ_n(kr) + BJ_{-n}(kr)\}] \sin\theta \, e^{ipt}.$$

Using the recurrence formulae

$$J'_n(z) = \frac{n}{z} J_n(z) - J_{n+1}(z)$$

$$J'_{-n}(z) = J_{-n-1}(z) + \frac{n}{z} J_{-n}(z),$$

we get

$$\widehat{\varphi r} = c_{55} r^{-3/2}[A\{\tfrac{1}{2}(2n-3)J_n(kr) - krJ_{n+1}(kr)\} + B\{\tfrac{1}{2}(2n-3)J_{-n}(kr) + krJ_{-n-1}(kr)\}] \sin\theta \, e^{ipt}. \tag{21}$$

Putting $n = 3/2$, we get

$$\widehat{\varphi r} = kc_{55} r^{-\frac{1}{2}}[-AJ_{5/2}(kr) + BJ_{-5/2}(kr)] \sin\theta \, e^{ipt}.$$

Since $\widehat{\varphi r} = 0$ when $r = a$ and $r = b$, we have

$$-AJ_{5/2}(ka) + BJ_{-5/2}(ka) = 0 \tag{22}$$

$$-AJ_{5/2}(kb) + BJ_{-5/2}(kr) = 0 \tag{23}$$

Eliminating A and B from (22) and (23) we get the frequency equation as

$$\frac{J_{5/2}(ka)}{J_{-5/2}(ka)} = \frac{J_{5/2}(kb)}{J_{-5/2}(kb)}. \tag{24}$$

We can simplify this equation by using the relations

$$J_{5/2}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \left[\sin z \left(\frac{3}{z^2} - 1\right) - \frac{3 \cos z}{z} \right]$$

$$J_{-5/2}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \left[\cos z \left(\frac{3}{z^2} - 1\right) + \frac{3 \sin z}{z} \right].$$

In conclusion I express my gratefulness to Dr. S. Ghosh of University College of Science for his valuable guidance in preparing this paper.

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References

Saint-Venant, (1865), *J. Math.* (Liouville), Sér. 2, 10.

Love, A. E. H. (1944), *A treatise on the mathematical theory of elasticity*, 4th. ed. Dover Publications.

Seth, B. R. (1946), *Bull. Cal. Math. Soc.* 38, 39.

ON SYMBOLIC CALCULUS OF TWO VARIABLES

By

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(*Received—October 14, 1953*).

1. The object of the present paper is to derive some theorems in Symbolic Calculus of two variables as developed by Mclachlan and Humbert (1949), Delerue (1951), Chakravarty (1953) and Voelker and Doetsch (1950). We shall use the symbolic notation $f(p, q) \subset\subset h(x, y)$ or $h(xy) \supset\supset f(p, q)$ to represent the convergent double integral,

$$\frac{f(p, q)}{pq} = \int_0^\infty \int_0^\infty e^{-px-qy} h(x, y)\,dx\,dy,\ R(p)>0,\ R(q)>0.$$

In the theorems that follow we have derived symbolic connections in two variables of the types $f(x/y) \supset\supset \varphi(q/p)$ or $f(xy) \supset\supset \varphi(pq)$ when f and φ are connected by a chain of symbolic relations in one variable. The theorems have been illustrated by suitable examples.

2. Theorem I. *If* $f(p) \subset h(x)$ *and* $(1/p^m)h(p) \subset g(x)$, *then* $x^m g(y/x) \supset\supset q^{-m} f(p/q)$.

Proof. Since $f(p) \subset h(x)$ and $(1/p^m)h(p) \subset g(x)$, we have

$$f(p) = \Gamma(m+2)p \int_0^\infty \frac{g(t)}{(p+t)^{m+2}}dt,\ R(m)>-2.$$

(Shastri, p. 212),

so that

$$f(p/q) = \Gamma(m+2)pq^{m+1} \int_0^\infty \frac{g(t)\,dt}{(p+qt)^{m+2}} \tag{2.1}$$

Now

$$\Psi(p, q) = pq \int_0^\infty e^{-px-qy} \theta(y/x) y^\mu\,dx\,dy \tag{2.2}$$

Substituting $y = tx$ and then integrating with respect to x we obtain

$$\Psi(p, q) = pq\Gamma(\mu+2) \int_0^\infty \frac{t^\mu \theta(t)dt}{(p+qt)^{\mu+2}} \tag{2.3}$$

Putting $\mu = m$ and $t^m\theta(t) = g(t)$, we have

$$y^m \theta(y/x) = x^m g(y/x). \tag{2.4}$$

Hence from (2.1)—(2.4), we obtain

$$q^{-m} f(p/q) \subset\subset x^m g(y/x),$$

which establishes the theorem, valid when $R(m)>-2$.

It may noted that this theorem occurs in *Memorial des Sci. Math.* fasc. 127, published in 1954.

Cor. I Put $m = 0$ and we obtain the theorem due to Delerue (1951), *viz.*,

"*If* $f(p) \subset h(x)$ *and* $h(p) \subset g(x)$, *then* $f(p/q) \subset\subset g(y/x)$."

Cor. II *Let* $\varphi(p) \subset H(x)$; *Then* $h(p) \equiv \dfrac{\varphi(p)}{p} \subset \displaystyle\int_0^x H(t)dt \equiv g(x)$,

and
$$h(x) \equiv \frac{\varphi(x)}{x} \supset p \int_0^\infty \frac{H(t)dt}{p+t} \equiv f(p), \text{ say,}$$

Hence, by cor I, we obtain

$$\int_0^{y/x} H(t)dt \supset\supset p \int_0^\infty \frac{H(t)dt}{p+qt} \tag{2.5}$$

provided that the integrals involved are convergent.

To illustrate (2.5), we have the operational representation

$$x^{\frac{1}{2}(\nu+1)} J_{\nu+1}(2x^{\frac{1}{2}}) \supset \frac{1}{p^{\nu+1}} e^{-1/p} \equiv \frac{1}{p} \cdot \frac{1}{p^\nu} e^{-1/p}$$

so that the product theorem gives

$$x^{\frac{1}{2}(\nu+1)} J_{\nu+1}(2x^{\frac{1}{2}}) = \int_0^x x^{\frac{1}{2}\nu} J_\nu(2x^{\frac{1}{2}}) dx.$$

Hence by (2.5)

$$(y/x)^{\frac{1}{2}(\nu+1)} J_{\nu+1}(2y^{\frac{1}{2}}/x^{\frac{1}{2}}) \supset\supset p \int^\infty \frac{x^{\frac{1}{2}\nu} J_\nu(2x^{\frac{1}{2}}) dx}{p+qx}.$$

Evaluating the R. H. side integral by Watson, p. 434, (1952), we are lead to the operational representation

$$(y/x)^{\frac{1}{2}(\nu+1)} J_{\nu+1}(2y^{\frac{1}{2}}/x^{\frac{1}{2}}) \supset\supset 2(p/q)^{\frac{1}{2}(\nu+1)} K_\nu(2p^{\frac{1}{2}}/q^{\frac{1}{2}}),$$

a result already given by Delerue (1951).

Secondly, we have the operational representation

$$e^{-x} x^m T_m^n(x) \supset \frac{(-1)^n}{\Gamma(n+1)} \frac{p^{n+1}}{(1+p)^{m+n+1}},$$

where $T_m^n(x)$ is Sonine's polynomial.

By the product theorem

$$\int_0^x e^{-x} x^m T_m^n(x) dx \supset \frac{(-1)^n}{\Gamma(n+1)} \cdot \frac{p^n}{(1+p)^{m+n+1}} = -\frac{1}{n} e^{-x} x^{m+1} T_{m+1}^{n-1}(x),$$

and by Shastri (1944)

$$\int_0^\infty \frac{e^{-t} t^m T_m^n(t)\, dt}{(p+t)} = (-1)^n e^{\frac{1}{2}p} p^{\frac{1}{2}(-1+m)} W_{-\frac{1}{2}(m+2n+1),\, -\frac{1}{2}m}(p)\, R(n+1) \geq 0.$$

Hence (2.5) leads to the operational repesentation

$$e^{-y/x}(y/x)^{m+1}T^{n}_{m+1}(y/x)\supset\supset n(-1)^{n+1}e^{\frac{1}{2}p/q}(p/q)^{\frac{1}{2}(m+1)}W_{\frac{1}{2}(m+2n+1),\,-\frac{1}{2}m}(p/q) \qquad (2.6)$$

In the following we shall employ theorem I to obtain some new operational representations.

(1) Let $f(p) = -\gamma - \log p$; so that $h(x) = \log x$; and by Mclachlan N. W (1949, p. 116),

$$x^{m} \log x \supset \frac{\Gamma(1+m)}{p^{m}}[\psi(1+m) - \log p],\ \mathrm{R}(m) > -1,$$

and

$$\psi = \Gamma'/\Gamma = \frac{d}{dm} \log \Gamma(1+m).$$

from which

$$(1/p^{m}) \log p \supset \frac{x^{m}}{\Gamma(1+m)}[\psi(1+m) - \log x] \equiv g(x), \quad \text{say.}$$

Hence we obtain the operational representation

$$\frac{y^{m}}{\Gamma(1+m)}[\psi(1+m) - \log (y/x)] \supset\supset q^{-m}[-\gamma - \log (p/q)] \qquad (2.7)$$

(2) By Mclachlan N. W. (1949, p. 119), we have

$$x^{\frac{1}{2}\nu} \operatorname{ber}_{\nu} x^{\frac{1}{2}} \supset [1/(2p)^{\nu}] \cos (1/4p + \tfrac{3}{4}\nu\pi)$$

and

$$x^{\frac{1}{2}\nu} \operatorname{bei}_{\nu} x^{\frac{1}{2}} \supset [1/(2p)^{\nu}] \sin (1/4p + \tfrac{3}{4}\nu\pi),$$

so that

$$(1/p^{\nu}) \cos (1/4p) \subset 2^{\nu}x^{\frac{1}{2}\nu}[\operatorname{ber}_{\nu} x^{\frac{1}{2}} \cos \tfrac{3}{4}\nu\pi + \operatorname{bei}_{\nu} x^{\frac{1}{2}} \sin \tfrac{3}{4}\nu\pi] \qquad (2.8)$$

and

$$(1/p^{\nu}) \sin (1/4p) \subset 2^{\nu}x^{\frac{1}{2}\nu}[\operatorname{bei}_{\nu} x^{\frac{1}{2}} \cos \tfrac{3}{4}\nu\pi] - \operatorname{ber}_{\nu} x^{\frac{1}{2}} \sin \tfrac{3}{4}\nu\pi] \qquad (2.9)$$

Again by Mclachlan, (1949, p. 127), we have

$$\frac{1}{x^{\frac{1}{2}}} \cos \frac{1}{4x} \supset (\pi p)^{\frac{1}{2}} e^{-(p/2)^{\frac{1}{2}}} \cos(p/2)^{\frac{1}{2}} \qquad (2.10)$$

and

$$\frac{1}{x^{\frac{1}{2}}} \sin \frac{1}{4x} \supset (\pi p)^{\frac{1}{2}} e^{-(p/2)^{\frac{1}{2}}} \sin(p/2)^{\frac{1}{2}} \qquad (2.11)$$

We then obtain the following two opertional representations:

$$\frac{(p\pi)^{\frac{1}{2}}}{q^{\nu}} e^{-(p/2q)^{\frac{1}{2}}} \cos(p/2q) \subset\subset 2^{\nu}x^{\frac{1}{2}(\nu-1)}y^{\frac{1}{2}\nu}\left[\cos \frac{3\nu\pi}{4} \operatorname{ber}_{\nu}\left(\frac{y}{x}\right)^{\frac{1}{2}} + \sin \frac{3\nu\pi}{4} \operatorname{bei}_{\nu}(y/x)\right] \qquad (2.12)$$

and

$$\frac{(p\pi)^{\frac{1}{2}}}{q^{\nu}} e^{-(p/2q)^{\frac{1}{2}}} \sin\left(\frac{p}{2q}\right)^{\frac{1}{2}} \subset\subset 2^{\nu}x^{\frac{1}{2}(\nu-1)}y^{\frac{1}{2}\nu}\left[\cos \frac{3\nu\pi}{4} \operatorname{bei}_{\nu}\left(\frac{y}{x}\right)^{\frac{1}{2}} - \sin \frac{3\nu\pi}{4} \operatorname{ber}_{\nu}\left(\frac{y}{x}\right)^{\frac{1}{2}}\right]. \qquad (2.13)$$

and in particular,

$$(p\pi)^{\frac{1}{2}} e^{-(p/2q)} \begin{matrix}\cos \\ \sin\end{matrix} \left(\frac{p}{2q}\right)^{\frac{1}{2}} \subset\subset \frac{1}{x^{\frac{1}{2}}} \begin{matrix}\operatorname{ber} \\ \operatorname{bei}\end{matrix} \left(\frac{y}{x}\right)^{\frac{1}{2}} \qquad (2.13\text{ a})$$

(3) N. A. Shastri (1944) has obtained the following operational representation:

$$(2/x^{k+\frac{1}{2}})K_{2m}(2x^{\frac{1}{2}}) \supset \Gamma(\tfrac{1}{2}+m-k)\Gamma(\tfrac{1}{2}-m-k)e^{1/(2p)}p^{k+1}W_{k,m}(1/p)\ R(\tfrac{1}{2}\pm m-k)>0.$$

Further, we have

$$2p^{1-m}K_{2m}(2p^{\frac{1}{2}})\subset x^{2m-1}e^{-1/x},\ R(p)>0$$

So we obtain the following operational representation

$$x^{-m-k-\frac{1}{2}}y^{2m-1}e^{-x/y}\supset\supset\Gamma(\tfrac{1}{2}+m-k)\Gamma(\tfrac{1}{2}-m-k)p^{k+1}q^{-m+\frac{1}{2}}e^{q/2p}W_{k,m}(q/p),$$
$$R(\pm m-k+\tfrac{1}{2})>0. \quad (2.14)$$

The following special cases are worth notice:

Since,

$$S(\nu, z) = z^{\frac{1}{2}\nu-1}e^{\frac{1}{2}z}W_{-\frac{1}{2}\nu,\frac{1}{2}-\frac{1}{2}\nu}(z),\ \nu>0.$$

We obtain

$$x^{\nu-1}y^{-\nu}e^{-x/y}\supset\supset\Gamma(\nu)qS(\nu, q/p) \quad (2.15)$$

Since $$\mathrm{Ei}(z)\equiv\int_z^\infty\frac{e^{-t}}{t}\,dt = e^{-z}S(1, z).$$

(Whittaker and Watson 1940, p. 352),
we obtain

$$\frac{1}{y}e^{-x/y}\supset\supset qe^{q/p}\mathrm{Ei}\left(\frac{q}{p}\right) = -qe^{q/p}\,\mathrm{li}(e^{-q/p}) \quad (2.16)$$

$\mathrm{li}(e^{x})$ having the definition as given in Whittaker and Watson, p. 342.

Again, since $$\mathrm{Erfc}\,(x) = \tfrac{1}{2}x^{-\frac{1}{2}}e^{-\frac{1}{2}x^2}W_{-\frac{1}{4},\frac{1}{4}}(x^2),$$

(Whittaker and Watson 1940, p. 348), we obtain, from (2.16)

$$[1/(xy)^{\frac{1}{2}}]\,e^{-x/y}\supset\supset 2(\pi pq)^{\frac{1}{2}}e^{q/p}\mathrm{Erfc}(q/p). \quad (2.17)$$

(4) We have the operational representations

$$(1/x^{2\mu+1})\ e^{-1/x^2}\supset 2\pi^{3/2}(\tfrac{1}{2}p)^{\frac{2}{3}(2+\mu)}\mathrm{cosec}\,\mu\pi\big[J_{-\mu,-\frac{1}{3}}(3(\tfrac{1}{2}p)^{\frac{2}{3}})\cos\mu\pi$$
$$+J_{\frac{1}{3}-\mu,\frac{1}{3}}(3(\tfrac{1}{2}p)^{\frac{2}{3}})\sin\mu\pi-J_{\mu-\frac{1}{3},\frac{1}{3}}(3(\tfrac{1}{2}p)^{\frac{2}{3}})\big] \quad (2.18)$$

(Gupta, H. C., 1948, p. 149).

Also $$\frac{1}{p^{m+2\mu+1}}\,e^{-1/p^2}\subset x^{m+2\mu+1}J^2_{m+2\mu+1}(x^2)$$

where $J^{\mu}_{\lambda}(x)$ is the Wright's generalised Bessel function

Hence the theorem gives the operational representation

$$x^{-2\mu-1}y^{m+2\mu+1}J^2{}_{m+2\mu+1}(y^2/x^2)\supset\supset 2^{-\frac{1}{3}(1+2\mu)}\pi^{3/2}p^{\frac{2}{3}(2+\mu)}q^{-m-\frac{2}{3}(2+\mu)}\ \mathrm{cosec}\,2\mu\pi$$
$$\times\left\{J_{-\mu,-\frac{1}{3}}\left[3\left(\frac{p^2}{4q^2}\right)^{\frac{1}{3}}\right]\cos\mu\pi+J_{\frac{1}{3}-\mu,\frac{1}{3}}\left[3\left(\frac{p^2}{4q^2}\right)^{\frac{1}{3}}\right]\sin\mu\pi-J_{\mu-\frac{1}{3},\frac{1}{3}}\left[3\left(\frac{p^2}{4q^2}\right)^{\frac{1}{3}}\right]\right\}, \quad (2.19)$$

$R(m)>-2.$

(5) We have the operational representation

$$\Gamma(\mu+\nu)\frac{p^{1-\nu+\rho}}{(1+p)^{\rho+\mu}} \subset x^{\frac{1}{2}\mu+\frac{1}{2}\nu-1}e^{-\frac{1}{2}x}M_{\rho+\frac{1}{2}\mu-\frac{1}{2}\nu,\frac{1}{2}(\mu+\nu-1)}(x), \quad R(\mu+\nu)>0,$$

(Erdelyi (1936), p. 131),

and $$\frac{x^{\rho+1}}{(1+x)^{\rho+\mu}} \supset \frac{\Gamma(\varrho+2)}{p^{\frac{1}{2}(1-\mu)}}\, e^{\frac{1}{2}p}W_{-\frac{1}{2}(\mu+2\rho+1),\frac{1}{2}(2-\mu)}(p), \quad R(\varrho+2)>0,$$

a result easily deducible from Whittaker's integral for $W_{k,m}(x)$.

Hence the theorem leads to the operational relation

$$x^{\frac{1}{2}\nu-\frac{1}{2}\mu+1}y^{\frac{1}{2}\nu+\frac{1}{2}\mu-1}e^{-\frac{1}{2}y/x}M_{\rho+\frac{1}{2}\mu-\frac{1}{2}\nu,\frac{1}{2}(\mu+\nu-1)}(y/x)$$
$$\supset\supset\Gamma(\mu+\nu)\Gamma(\varrho+2)p^{\frac{1}{2}(\mu-1)}q^{\frac{1}{2}(1-\mu-2\nu)}e^{\frac{1}{2}p/q}\,W_{-\frac{1}{2}(\mu+2\rho+1),\frac{1}{2}(2-\mu)}(p/q) \tag{2.22}$$

(6) We have $$\frac{x^2}{(1+x^2)^{\mu+\frac{1}{2}}} = \frac{1}{(1+x^2)^{\mu-\frac{1}{2}}} - \frac{1}{(1+x^2)^{\mu+\frac{1}{2}}}$$

$$\supset \tfrac{1}{2}\pi^{\frac{1}{2}}\Gamma(\tfrac{1}{2}-\mu)(\tfrac{1}{2}p)^{\mu}[(1-2\mu)\{H_{1-\mu}(p)-Y_{1-\mu}(p)\}-p\{H_{-\mu}(p)-Y_{-\mu}(p)\}],$$

$\frac{1}{2}<R(\mu)<3/2.$

by Mclachlan and Humbert (1941),

and $$\frac{p}{(1+p^2)^{\mu+\frac{1}{2}}} \subset \frac{\pi^{\frac{1}{2}}}{2^{\mu}\Gamma(\mu+\frac{1}{2})}\, x^{\mu}J_{\mu}(x), \ (\mu>-\tfrac{1}{2}).$$

Hence we obtain the operational representation

$$x^{1-\mu}y^{\mu}J_{\mu}(y/x)$$

$$\supset\supset \frac{\pi p^{\mu}}{2\cos\mu\pi.\, q^{\mu+1}}\left[(1-2\mu)\left\{H_{1-\mu}\left(\frac{p}{q}\right)-Y_{1-\mu}\left(\frac{p}{q}\right)\right\}-\frac{p}{q}\left\{H_{-\mu}\left(\frac{p}{q}\right)-Y_{-\mu}\left(\frac{p}{q}\right)\right\}\right] \tag{2.21}$$

3. Theorem II. *If* $f(p)\subset h(x)$, $\frac{1}{p^{s-1}}h\left(\frac{1}{p}\right)\supset g(x)$, $\frac{1}{p^{s-1}}g(p)\subset\varphi(x)$ *and* $x^{s-1}\varphi\left(\frac{1}{x}\right)\supset H(p)$, *then* $(x/y)^{s-1}H(x/y)\supset\supset f(q/p)$.

Proof. Since $f(p)\subset h(x)$ and $\frac{1}{p^{s-1}}h\left(\frac{1}{p}\right)\subset g(x)$, we have by Shastri (1944)

$$f(p) = 2p^{\frac{1}{2}(s+1)}\int_0^{\infty} t^{-\frac{1}{2}(s-1)}g(t)K_{s-1}(2(pt)^{\frac{1}{2}})dt \tag{A}$$

which can be written in the form

$$f\left(\frac{q}{p}\right) = \int_0^{\infty} t^{-s}g(t)dt.2\left(\frac{qt}{p}\right)^{\frac{1}{2}(s+1)} K_{s-1}(2(qt/p)^{\frac{1}{2}}) \tag{3.1}$$

Interpreting (3.1) by

$$(x/y)^{\frac{1}{2}(\nu+1)}J_{\nu+1}(2(x/y)^{\frac{1}{2}})\supset\supset 2(q/p)^{\frac{1}{2}(\nu+2)}K_{\nu}(2(q/p)^{\frac{1}{2}})$$

we obtain, on simplification,

$$f(q/p) \subset\subset (x/y)^{\frac{1}{2}s} \int_0^\infty t^{-\frac{1}{2}s} J_s(2(xt/y)^{\frac{1}{2}}) g(t)\, dt \tag{3.2}$$

Again, since $\frac{1}{p^{s-1}} g(p) \subset \varphi(x)$, we have, by Gupta (1948, p. 142)

$$x^{s-1}\varphi(1/x) \supset p^{1-\frac{1}{2}s} \int_0^\infty t^{-\frac{1}{2}s} g(t) J_s(2(pt)^{\frac{1}{2}})\, dt \tag{3.3}$$

which, by $x^{s-1}\varphi\left(\frac{1}{x}\right) \supset H(p)$, can be written as

$$H(p) = p^{1-\frac{1}{2}s} \int_0^\infty t^{-\frac{1}{2}s} g(t) J_s(2(pt)^{\frac{1}{2}})\, dt.$$

So, $$(x/y)^{\frac{1}{2}s-1} H\left(\frac{x}{y}\right) = \int_0^\infty t^{-\frac{1}{2}s} g(t) J_s\left(2\left(\frac{xt}{y}\right)^{\frac{1}{2}}\right) dt \tag{3.4}$$

From (3.2) and (3.4), we obtain

$$f(q/p) \subset\subset (x/y)^{s-1} H(x/y),$$ which is the theorem.

For the validity of the theorem, we first ensure the existence of (A). For this we should have $g(y)$ bounded and absolutely integrable in $(0, \infty)$ and $g(y) = O(y^{s-1+\delta})$, when y is small and $\delta > 0$.

Again, the integral on the right of (3.2) or (3.4) is convergent if $R(s) > \delta > 0$ and $g(x)$ is bounded and absolutely integrable in $(0, \infty)$. Hence, for the validity of the theorem, we should have

(i) $g(x)$ bounded and absolutely in $(0, \infty)$

(ii) $s > \delta > 0$. It is enough if $s > 1$.

Cor. I. Put $s = 3/2$ and utilise the result that

"*If* $f(p) \subset h(x)$ *and* $(1/p^{\frac{1}{2}}) h(1/p) \subset g(x)$, *then* $(1/p) f(p^2/4) \subset \frac{1}{2}\pi^{\frac{1}{2}} g(x^2)$."

The theorem, then, can be stated in the form

"*If* $(2/\pi^{\frac{1}{2}})(1/p) f(p^2/4) \subset g(x^2)$, $(1/p^{\frac{1}{2}}) g(p) \subset \varphi(x)$ *and* $x^{\frac{1}{2}}\varphi(1/x) \supset H(p)$, *then* $(x/y)^{\frac{1}{2}} H(x/y) \supset\supset f(q/p)$."

As an application,

let $x^{\frac{1}{2}}\varphi(1/x) = x^{s-1} e^{-1/x}$; so that $H(p) = 2p^{1-\frac{1}{2}s} K_s(2p^{\frac{1}{2}})$, $R(p) > 0$,

and $\varphi(x) = x^{3/2-s} e^{-x}$, and hence $g(p) = \Gamma(5/2-s) \dfrac{p^{3/2}}{(1+p)^{5/2-s}}$.

Now,

$$\frac{x^3}{(1+x^2)^{5/2-s}} \supset \frac{\Gamma(\frac{1}{2}-s)}{2\Gamma(5/2-s)}\, p\, {}_1F_2\left(\begin{matrix}2\\ \frac{1}{2}, \frac{1}{2}+s\end{matrix}; -\tfrac{1}{4}p^2\right) - \frac{\pi^{\frac{1}{2}}.3\Gamma(-s)}{8\Gamma(5/2-s)}\, p^2\, {}_1F_2\left(\begin{matrix}5/2\\ 3/2, 1+s\end{matrix}; -\tfrac{1}{4}p^2\right)$$

$$+\Gamma(2s-1) p^{2-2s} {}_1F_2\left(\begin{matrix}5/2-s\\ 1-s, 3/2-s\end{matrix}; -\tfrac{1}{4}p^2\right),$$

by a result due to H. C. Gupta. (1948, p. 149), *viz.*,

$$\frac{x^{2\rho}}{(1+x)^{\mu+\rho+\frac{1}{2}}} \supset \frac{\Gamma(\varrho+\frac{1}{2})\Gamma(\mu)}{2\Gamma(\mu+\varrho+\frac{1}{2})}\, p\; {}_1F_2\left({\varrho+\frac{1}{2} \atop \frac{1}{2},\, 1-\mu}; -\frac{p^2}{4}\right) - \frac{\Gamma(\varrho+1)\Gamma(\mu-\frac{1}{2})}{2\Gamma(\varrho+\mu+\frac{1}{2})}\, p^2\; {}_1F_2\left({\varrho+1 \atop \frac{3}{2},\, \frac{3}{2}-\mu}; -\frac{p^2}{4}\right)$$

$$+\Gamma(-2\mu)p^{1+2\mu}{}_1F_2\left({\mu+\varrho+\frac{1}{2} \atop \mu+\frac{1}{2},\, \mu+1}; -\tfrac{1}{4}p^2\right)$$

Thus $f(p)$ is given by

$$\frac{2}{\pi^{\frac{1}{2}}}f(p) = 2\Gamma(\tfrac{1}{2}-s)p\; {}_1F_2\left({2 \atop \frac{1}{2},\, \frac{1}{2}+s}, -p\right) - 3\pi^{\frac{1}{2}}\Gamma(-s)p^{3/2}{}_1F_2\left({5/2 \atop 3/2,\, 1+s}; -p\right)$$

$$+\Gamma(2s-1)\Gamma(5/2-s)4^{3/2-s}p^{3/2-s}{}_1F_2\left({5/2-s \atop 1-s,\, 3/2-s}; -p\right),$$

and the theorem gives the operational relation

$$\frac{4}{\pi^{\frac{1}{2}}}\left(\frac{x}{y}\right)^{\frac{1}{2}(3-s)}K_s(2(x/y)^{\frac{1}{2}}) \supset\supset 2\Gamma(\tfrac{1}{2}-s)(q/p)\; {}_1F_2\left({2 \atop \frac{1}{2},\, \frac{1}{2}+s}; -\frac{q}{p}\right)$$

$$-3\pi^{\frac{1}{2}}\Gamma(-s)(q/p)^{3/2}{}_1F_2\left({5/2 \atop 3/2,\, 1+s}; -\frac{q}{p}\right)$$

$$+\Gamma(2s-1)\Gamma\left(\frac{5}{2}-s\right)4^{3/2-s}(q/p)^{3/2-s}{}_1F_2\left({5/2-s \atop 1-s,\, 3/2-s}; -\frac{q}{p}\right) \tag{3.5}$$

4. Theorem III *If* $H(p)\subset x^{\mu-1}\psi(x)$, $\psi(p)\subset x^{2\lambda-1}f(x)$, $f(p)\subset\varphi(x)$ *and* $x^{2\lambda-1}\varphi(1/x)$ $\supset h(p)$, *then* $x^{\mu-1}y^{2\lambda-1}h(xy)\supset\supset q^{\mu-2\lambda}H(pq)$.

Proof. Since $f(p)\subset\varphi(x)$, therefore by Gupta H. C. (1948)

$$x^{2\lambda-1}\varphi\left(\frac{1}{x}\right)\supset p^{1-\lambda}\int_0^\infty t^{\lambda-1}J_{2\lambda}(2(pt)^{\frac{1}{2}})f(t)\,dt \tag{4.1}$$

which, by $x^{2\lambda-1}\varphi(1/x)\supset h(p)$, can be written as $h(p) = p^{1-\lambda}\int_0^\infty t^{\lambda-1}J_{2\lambda}(2(pt)^{\frac{1}{2}})f(t)dt$.

Writing the above in the form

$$\frac{x^{\mu-1}y^{2\lambda-1}}{\Gamma(\mu+1)}h(xy) = \int_0^\infty t^{-1}f(t)dt\frac{x^{\mu-\lambda}(yt)^\lambda}{\Gamma(\mu+1)}J_{2\lambda}(2(xyt)^{\frac{1}{2}})$$

and interpreting by the help of operational representation

$$\frac{x^{\mu-\lambda}y^\lambda}{\Gamma(\mu+1)}J_{2\lambda}(2(xy)^{\frac{1}{2}})\supset\supset\frac{pq^{\mu-2\lambda+1}}{(1+pq)^{\mu+1}}$$

we obtain,

$$\frac{x^{\mu-1}y^{2\lambda-1}h(xy)}{\Gamma(\mu+1)}\supset\supset pq^{\mu-2\lambda+1}\int_0^\infty\frac{t^{2\lambda-1}f(t)dt}{(t+pq)^{\mu+1}} \tag{4.2}$$

Since
$$\psi(p) \subset x^{2\lambda-1}f(x),$$
and
$$e^{-ax}x^{\mu} \supset \Gamma(\mu+1)\frac{p}{(p+a)^{\mu+1}},\ R(\mu)>-1,$$
so that
$$x^{\mu-1}\psi(x) \supset \Gamma(\mu+1)p\int_0^{\infty}\frac{t^{2\lambda-1}f(t)dt}{(t+p)^{\mu+1}} \tag{4.3}$$
Since
$$x^{\mu-1}\psi(x) \supset H(p),$$
we have, from (4.3)
$$\frac{H(pq)}{\Gamma(\mu+1)pq} = \int_0^{\infty}\frac{t^{2\lambda-1}f(t)dt}{(t+pq)^{\mu+1}}. \tag{4.4}$$

The theorem now follows from (4.2) and (4.4).

A set of conditions for the validity of the theorem may be as follows:

(*i*) $f(t)$ bounded and absolutely integrable in $(0, \infty)$

(*ii*) $0<\lambda<5/4$

(*iii*) $\mu+1>2\lambda$.

Example I. Let $f(p) = \log p$; so that $\varphi(x) = -\gamma-\log x$.

Then since
$$x^{\nu}\log x \supset \frac{\Gamma(1+\nu)}{p^{\nu}}\ [\psi(1+\nu)-\log p],\ R(\nu)>-1,$$
we have
$$h(p) = \frac{\Gamma(2\lambda)}{p^{2\lambda-1}}\ [-\gamma+\psi(2\lambda)-\log p],\ \psi \equiv \frac{\Gamma'}{\Gamma}.$$
and
$$H(p) = \frac{\Gamma(2\lambda)\Gamma(\mu-2\lambda+1)}{p^{\mu-2\lambda}}[\psi(2\lambda)-\psi(\mu-2\lambda+1)+\log p].$$

Hence the theorem gives the operational representation
$$x^{\mu-2\lambda}[-\gamma+\psi(2\lambda)-\log xy]$$
$$\supset\supset \frac{\Gamma(\mu-2\lambda+1)}{p^{\mu-2\lambda}}\ [\psi(2\lambda)-\psi(\mu-2\lambda+1)+\log pq],\ R(\mu-2\lambda)>-1 \tag{4.5}$$

In particular,
$$x^{\mu-1}[-\gamma-\log xy] \supset\supset \frac{\Gamma(\mu)}{p^{\mu-1}}[-\psi(\mu)+\log pq] \tag{4.6}$$

Example II. Let
$$f(p) = \Gamma(\nu+1)\frac{p}{(1+p)^{\nu+1}};\ \text{so that } \varphi(x) = x^{\nu}e^{-x},$$
therefore
$$x^{2\lambda-1}f(x) = \Gamma(\nu+1)\frac{x^{2\lambda}}{(1-x)^{\nu+1}},\ R(\nu)>0,$$
$$\supset \Gamma(\nu+1)\Gamma(2\lambda+1)p^{\frac{1}{2}(\nu-2\lambda+1)}e^{\frac{1}{2}p}W_{-\frac{1}{2}(2\lambda+\nu+1),\,-\frac{1}{2}(\nu-2\lambda)}(p)$$
$$= \Gamma(\tfrac{1}{2}-m-k)\Gamma(\tfrac{1}{2}+m-k)p^{l-\mu}e^{\frac{1}{2}p}W_{k,m}(p) \equiv \psi(p),$$

where $\qquad l = \mu+\frac{1}{2}(\nu-2\lambda+1),\ k = -\frac{1}{2}(2\lambda+\nu+1),\ m = -\frac{1}{2}(\nu-2\lambda).$

Hence $\qquad x^{\mu-1}\psi(x) = \Gamma(\frac{1}{2}-m-k)\Gamma(\frac{1}{2}+m-k)x^{l-1}e^{\frac{1}{2}x}W_{k,m}(x)$

$$\supset\frac{\Gamma(\frac{1}{2}-m-k)\Gamma(\frac{1}{2}+m-k)\Gamma(l+m+\frac{1}{2})\Gamma(l-m+\frac{1}{2})}{\Gamma(l-k+1)}\times$$

$$p\ {}_2F_1\left(\begin{matrix} l+m+\frac{1}{2},\ l-m+\frac{1}{2} \\ l-k+1\end{matrix};\ 1-p\right)\equiv H(p),\ \text{say}$$

by Goldstein (1932, p. 103).

and $\qquad x^{2\lambda-1}\phi(1/x) = x^{2\lambda-\nu-1}e^{-1/x}\equiv x^{2m-1}e^{-1/x}\supset 2p^{1-m}K_{2m}(2(p)^{\frac{1}{2}})\equiv h(p).$

We thus obtain the operational representation

$$2x^{l-\frac{1}{2}}y^{-k-\frac{1}{2}}K_{2m}(2(xy)^{\frac{1}{2}})\supset\supset\frac{\Gamma_2(\frac{1}{2}\mp m-k)\Gamma_2(l\pm m+\frac{1}{2})^*}{\Gamma(l-k+1)}\times$$

$$\times\, pq^{l+k+1}{}_2F_1\left(\begin{matrix} l+m+\frac{1}{2},\ l-m+\frac{1}{2} \\ l-k+1\end{matrix};\ 1-pq\right),\ l\pm m+\tfrac{1}{2}>0. \qquad (4.7)$$

The author is grateful to Dr. S. C. Mitra of Lucknow University for his keen interest and helpful criticisms during the preparation of the paper.

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References

Carslaw and Jaeger (1947), *Operational Methods in Applied Mathematics* (Second Edition),
Chakravarty N. K. (1953), *Ganita*, **4**, (1)
—————————, (1953), *Ann. de la Soc. Sci de Bruxelles*, Tome LXVII, I, mars.
Delerue P. (1951), Thesis for the D. Sc. degree of the University of Montpellier.
Erdelyi A. (1936), *Math Zeit.* **42**, 131,
Goldstein S. (1932), *Proc. Lond. Math Soc.* **34**, 103.
Gupta H. C (1948), *Proc Nat Inst Sci* (India) **14**, 131.
Mclachlan N. W. (1949), *Modern Operational Calculus.*
Mclachlan & Humbert (1941), *Memorial des Sci. Math. Fascicule* 100.
Shastri N. A. (1944), *Proc. Ind. Acad Sci* (A) **XX** 311.
Volker and Doetsch, (1950), *Die Zweidimensional Laplace-Transformation.*
Watson (1952), *Bessel functions* 434
Wright E. M. (1935), *Proc Lond Math. Soc* **38**, 257.
Whittaker and Watson (1940), *Course of Modern Analysis.*

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* The notation $\Gamma_2(\alpha\pm\beta+\gamma)$ means $\Gamma_2(\alpha\pm\beta+\gamma)=\Gamma(\alpha+\beta+\gamma)\Gamma(\alpha-\beta+\gamma)$.

AXISYMMETRIC FLOW IN PERFECT FLUID—II

MOTION OF A PARABOLOID OF REVOLUTION ALONG THE AXIS OF A ROTATING LIQUID

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(*Communicated by Prof B R. Seth—Received May 22, 1954*)

1. Introduction. The motion of a solid in perfect fluid endowed with vorticity is a problem of considerable interest but is unfortunately not very tractable. Even in the simplest case of a sphere moving along the axis of a rotating liquid, the restriction of small motion and consequent neglect of inertia terms seems to be necessary to obtain an approximate solution. However, in one case, when a sphere moves steadily along the axis a rotating liquid, Taylor (1922) has obtained exact solutions of the hydrodynamical equations. The problem is indeterminate because the boundary conditions of the perfect fluid theory are insufficient to determine all the arbitrary constants occuring in the solution. Long (1952) has generalised Taylor's solution and has shown that the general solution contains an infinite number of arbitrary constants even when all the boundary conditions are satisfied, and that Taylor's solution is a particular case of this general solution. Taylor thinks that different solutions represent stream lines due to different ways of starting the motion, while Long on the basis of the experiments, in which he observed waves behind the obstacle and no waves upstream thinks that the undetermined constants can be determined by adjusting the solution such that the upstream waves are completely anulled. The suggestion seems to be quite interesting but it is not quite clear as to how this could be done. The solution, however, can be made determinate by imposing an additional boundary condition that the liquid does not slip on the surface of the solid.

Nigam and Chatterji (1953) have shown that in the case of any body of revolution moving steadily along the axis of a rotating liquid, the general equations of motion in orthogonal coordinates reduce to a linear third order differential equation for the Stokes' stream function.

In another paper (Fadnis, 1955) exact solutions were obtained for the case of a spheroid, and a circular disc as the limiting case of the oblate spheroid. In this paper exact solutions have been obtained for a paraboloid of revolution moving along the axis of a rotating liquid. The solutions are obtained in terms of Whittaker functions and are made determinate after the manner of Taylor. It has not been possible to evaluate the constants numerically because the tables of these functions are not available.

2. Equations of motion. The equations governing the motion of a perfect incompressible fluid are

$$\mathbf{V} \times \omega = \text{grad } (p/\varrho + \mathbf{V}^2/2) \tag{1}$$

$$\text{div } \mathbf{V} = 0. \tag{2}$$

In what follows α, β denote the general orthogonal curvilinear coordinates in the meridian plane and γ the azimuthal angle; h_1, h_2, h_3, denote the elements of lenghths in the directions of α, β, γ increasing respectively, h_3 in addition represents the distance of any point from the axis of rotation. u, v, w denote the components of the velocity vector $\mathbf{V}$ and ξ', η', ζ' are the components of the vorticity vector $\boldsymbol{\omega}$ The motion being symmetrical about an axis all quantities are independent of the azimuthal angle γ. Equation (2) can be written as

$$\frac{\partial}{\partial \alpha}(h_2 h_3 u) + \frac{\partial}{\partial \beta}(h_1 h_3 v) = 0, \tag{3}$$

We introduce Stokes' stream function ψ such that

$$u = \frac{1}{h_2 h_3}\frac{\partial \psi}{\partial \beta}, \; v = -\frac{1}{h_1 h_3}\frac{\partial \psi}{\partial \alpha}, \; w = \frac{\Omega'}{h_3}. \tag{4}$$

With these values of u, v, w the components of vorticity are given by

$$\xi' = \frac{1}{h_2 h_3}\frac{\partial \Omega'}{\partial \beta}, \; \eta' = \frac{1}{h_1 h_3}\frac{\partial \Omega'}{\partial \alpha}, \; \zeta' = -\frac{1}{h_3} D^2 \psi \tag{5}$$

where

$$D^2 = \frac{h_3}{h_1 h_2}\left[\frac{\partial}{\alpha}\left(\frac{h_2}{h_1 h_3}\frac{\partial}{\partial \alpha}\right) + \frac{\partial}{\partial \beta}\left(\frac{h_1}{h_2 h_3}\frac{\partial}{\partial \beta}\right)\right]. \tag{6}$$

The third component of the vector equation of vorticity is

$$\frac{\partial}{\partial t}(D^2\psi) + \frac{2\Omega'}{h_1 h_2 h_3^2}\frac{\partial(\Omega', h_3)}{\partial(\alpha, \beta)} - \frac{1}{h_1 h_2 h_3}\frac{\partial(\psi, D^2\psi)}{\partial(\alpha, \beta)} + \frac{2}{h_1 h_2 h_3^2} D^2\psi \frac{\partial(\psi, h_3)}{\partial(\alpha, \beta)} = 0. \tag{7}$$

The third component of the vector equation of motion is

$$\frac{\partial \Omega'}{\partial t} + \frac{1}{h_1 h_2 h_3}\frac{\partial(\psi, \Omega')}{\partial(\alpha, \beta)} = 0 \tag{8}$$

For any body of revolution the above equations can be simplified in the following manner.

Let

$$h_3 w = \Omega' = K\psi \qquad [\psi_\infty] = -h_3^2\Omega_0/K. \tag{9}$$

Equation (8) is satisfied and (7) reduces to

$$\frac{D}{Dt}\left(\frac{\zeta'}{h_3} - \frac{K^2\psi}{h_3^2}\right) = 0 \tag{10}$$

or

$$\frac{\zeta'}{h_3} - \frac{K^2\psi}{h_3^2} = H(\psi) \tag{11}$$

Substuting for ζ' the equation for ψ is

$$(D^2+K^2)\psi = K\Omega_0 h_3^2 \tag{12}$$

whence
$$\psi = \frac{\Omega_0}{K}h_3^2 + A_1\psi_1 + A_2\psi_2 \tag{13}$$

where ψ_1 and ψ_2 are the appropriate solutions of

$$(D^2+K^2)\psi = 0. \tag{14}$$

3. Paraboloid of Revolution. We consider a paraboloid of revolution held at rest in a stream of liquid which is moving at infinity with a uniform velocity U_0 parallel to the axis of the paraboloid and rotating uniformly at infinity with a velocity Ω_0 about this axis. We then introduce a system of coordinates defined by

$$z+ir = (\beta+i\alpha)^2,\ \varphi = \gamma \tag{15}$$

where r, z are the cylindrical coordinates in the meridian plane and γ is the azimuth

The length elements are then respetively given by

$$h_1 = h_2 = 2\sqrt{(\alpha^2+\beta^2)},\ h_3 = 2\alpha\beta \tag{16}$$

and
$$D^2 = \frac{1}{4(\alpha^2+\beta^2)}\left[\frac{\partial^2\psi}{\partial\alpha^2} - \frac{1}{\alpha}\frac{\partial\psi}{\partial\alpha} + \frac{\partial^2\psi}{\partial\beta^2} - \frac{1}{\beta}\frac{\partial\psi}{\partial\beta}\right] \tag{17}$$

Equation (14) then reduces to

$$\frac{\partial^2\psi}{\partial\alpha^2} - \frac{1}{\alpha}\frac{\partial\psi}{\partial\alpha} + \frac{\partial^2\psi}{\partial\beta^2} - \frac{1}{\beta}\frac{\partial\psi}{\partial\beta} + 4K^2(\alpha^2+\beta^2)\psi = 0. \tag{18}$$

Putting $\psi = \psi_1(\alpha)\psi_2(\beta)$ equation (18) splits up into two separate equations.

$$\frac{\partial^2\psi_1}{\partial\alpha^2} - \frac{1}{\alpha}\frac{\partial\psi_1}{\partial\alpha} + (4K^2\alpha^2 - h)\psi_1 = 0 \tag{19}$$

$$\frac{\partial^2\psi_2}{\partial\beta^2} - \frac{1}{\beta}\frac{\partial\psi_2}{\partial\beta} + (4K^2\alpha^2 + h)\psi_2 = 0 \tag{20}$$

where 'h' is the constant of separation. Putting $\psi_1 = \alpha^2\varphi_1$ and $\alpha^2 = t$ (19) reduces to

$$t\frac{d^2\varphi_1}{dt^2} + 2\frac{d\varphi_1}{dt} + (K^2t - \tfrac{1}{4}h)\varphi_1 = 0 \tag{21}$$

putting $\psi_2 = \beta^2\varphi_2$, $\beta^2 = -\lambda$ equation (20) reduces to

$$\lambda\frac{d^2\varphi_2}{d\lambda^2} + 2\frac{d\varphi_2}{d\lambda} + (K^2\lambda - \tfrac{1}{4}h)\varphi_2 = 0 \tag{22}$$

Taking $h = -8iK(n+1)$ the differential equations (21) and (22) are satisfied by

$$\varphi_1 = e^{-iKt}F_1(2iKt)$$
$$\varphi_2 = e^{-iK\lambda}F_2 iK\lambda) \tag{23}$$

where
$$F_1(s),\ F_2(s),\ s = 2iKt,\ 2iK\lambda$$

are the solutions of the differential equation

$$s\frac{d^2F}{ds^2}+(2-s)\frac{dF}{ds}+nF=0. \tag{24}$$

We observe that equation (24) is a particular case of the differential equation satisfied by Laguerre polynomials

$$s\frac{d^2y}{ds^2}+(\mu+1-s)\frac{dy}{ds}+ny=0 \tag{25}$$

In our case μ has the value unity. This equation has got solutions one of which has a singularity at the origin If we designate our paraboloid $\alpha=\alpha_0$, for points on $\alpha=\alpha_0$, β varies from zero to infinity We also further observe that Laguerre polynomials can be very easily connected with Whittaker functions. Therefore for φ_1 we take the solution

$$1/t\,\{A_nW_{n+1,\frac{1}{2}}(2iKt)+B_nW_{-n-1,\frac{1}{2}}(-2iKt)\} \tag{26}$$

where W denotes a Whittaker function Remembering that $\psi_1=t\varphi_1=\alpha^2\varphi_1$ we get

$$\psi_1=\{A_nW_{n+1,\frac{1}{2}}(2iK\alpha^2)+B_nW_{-n-1,\frac{1}{2}}(-2iK\alpha^2)\}$$

For φ_2 we choose the solution $e^{-iK\lambda}L_n^{(1)}(2iK\lambda)$ where $L_n^{(1)}$ denotes a Laguerre polynomial defined by

$$L_n^{(1)}(s)=\sum_{p=0}^{\infty}\frac{(n+1)!}{(p+1)!(n-p)!}\,\frac{2^ps^p}{p!} \tag{27}$$

hence

$$\psi_2=\beta^2e^{iK\beta^2}L_n^{(1)}(-2iK\beta^2). \tag{28}$$

The complete expression for the stream function then becomes

$$\psi=\frac{4\Omega_0}{K}\alpha^2\beta^2+\sum_{n=0}^{\infty}\{A_nW_{n+1,\frac{1}{2}}(2iK\alpha^2)+B_nW_{-n-1,\frac{1}{2}}(-2iK\alpha^2)\}\beta^2e^{iK\beta^2}L_n^{(1)}(2iK\beta^2) \tag{29}$$

4. Expressions for the velocity components. On the surface of the paraboloid where $\alpha=\alpha_0$ we have

$$u=\frac{1}{h_2h_3}\frac{\partial\psi}{\partial\beta}=0 \quad\text{or}\quad \frac{\partial\psi}{\partial\beta}=0 \tag{30}$$

$$w=K\psi$$

The condition $\psi=0$ on $\alpha=\alpha_0$ makes the paraboloid a stream line. This therefore automatically satifies the condition of zero normal and zero rotational velocity on the boundary

If U_0 is the velocity of the liquid at infinity we find

$$\left[\frac{2\alpha U_0}{h_1}\right]_{\alpha=\infty}=\left\{\frac{1}{h_1h_3}\left[\frac{8\Omega_0\alpha^2\beta}{K}+\sum_{n=0}^{\infty}\{A_nW_{n+1,\frac{1}{2}}(2iK\alpha^2)+B_nW_{-n-1,\frac{1}{2}}(-2iK\alpha^2)\}\times\right.\right.$$

$$\left.\left.\frac{\partial}{\partial\beta}[\beta^2e^{iK\beta^2}L_n^{(1)}(-2iK\beta^2)]\right]\right\}_{\alpha=\infty} \tag{31}$$

Putting $\alpha = \infty$ we get $K = 2\Omega_0/U_0$

Making use of the fact that $\alpha = \alpha_0$ the paraboloid is a stream line we get

$$\frac{-4\Omega_0\alpha_0^2\beta^2}{K} = \sum_{n=0}^{\infty} \{A_n W_{n+1,\frac{1}{2}}(2iK\alpha^2) + B_n W_{-n-1,\frac{1}{2}}(-2iK\alpha^2)\}_{\alpha=\alpha_0}\beta^2 e^{iK\beta^2} L_n^{(1)}(-2iK\beta^2)$$

or $$\frac{-4\Omega_0\alpha_0^2 e^{iK\beta^2}}{K} = \sum_{n=0}^{\infty} \{A_n W_{n+1,\frac{1}{2}}(2iK\alpha^2) + B_n W_{-n-1,\frac{1}{2}}(-2iK\alpha^2)\}_{\alpha=\alpha_0} L_n^{(1)}(-2K\beta^2)$$

or $$-4\Omega_0\alpha_0^2\sum_0^{\infty}{}_p \frac{(-1)^p(iK\beta^2)^p}{p\,!} = \sum_{n=0}^{\infty}\left\{\sum_0^{\infty}{}_p \frac{(n+1)\,!\,2^p(iK\beta^2)^p}{(p+1)\,!\,(n-p)\,!\,p\,!}\right\}\chi_{1n} \tag{32}$$

where $$\chi_{1n} = \{A_n W_{n+1,\frac{1}{2}}(2iK\alpha_0^2) + B_n W_{-n-1,\frac{1}{2}}(-2iK\alpha_0^2).$$

Equating co-efficients of $(iK\beta^2)^p$ on both sides of (32) we have

$$\frac{(-1)^{p+1}4\Omega_0\alpha_0^2}{K} = \sum_n{}_p \frac{2^p(n+1)\,!}{(p+1)\,!\,(n-p)\,!}\,\chi_{1n}. \tag{33}$$

In this relation if we vary p from zero to infinity we get an infinite set of equations involving χ_{1n} where χ_{1n} are infinite sets of unknowns. It is observed that the number of unknowns in each successive equation is decreased by one. This fact enables us to solve these equations to any desired degree of accuracy After this is done we get a relation of the type

$$A_n W_{n+1,\frac{1}{2}}(2iK\alpha_0^2) + B_n W_{-n-1,\frac{1}{2}}(-2iK\alpha_0^2) = \int_n^{(1)} \tag{34}$$

where $\int_n^{(1)}$ is a known constant depending upon n. It is evident that by itself equation (34) is not sufficient to determine A_n and B_n uniquely. Therefore any values of A_n and B_n that satisfy (34) give a possible solution of the problem There are thus an infinite number of solutions consistent with the above boundary conditions. To make the problem determinate it is supposed that there is no slipping of the liquid on the surface of the paraboloid. This gives one more boundary condition, namely $\partial\psi/\partial\alpha = 0$ on $\alpha = \alpha_0$.

Making use of this boundary condition we get

$$\frac{-8\Omega_0\alpha_0 B^2}{K} = \sum_{n=0}^{\infty}\left\{A_n\frac{\partial}{\partial\alpha}W_{n+1,\frac{1}{2}}(2iK\alpha^2) + B_n\frac{\partial}{\partial\alpha}W_{-n-1,\frac{1}{2}}(-2iK\alpha^2)\right\}_{\alpha=\alpha_0} \times$$
$$\times\,\beta^2 e^{iK\beta^2} L_n^{(1)}(-2iK\beta^2)$$

$$\frac{8\Omega_0\alpha_0(-1)^{p+1}}{K}\sum_0^{\infty}{}_p\frac{(iK\beta^2)^p}{p\,!} = \sum_{n=0}^{\infty}\left\{\chi_{2n}\sum_0^{\infty}\frac{(n+1)\,!\,2^p(iK\beta^2)^p}{(p+1)\,!\,(n-p)\,!\,p\,!}\right\}$$

where $$\chi_{2n} = \left\{A_n\frac{\partial}{\partial\alpha}W_{n+1,\frac{1}{2}}(2iK\alpha^2) + B_n\frac{\partial}{\partial\alpha}W_{-n-1,\frac{1}{2}}(-2iK\alpha^2)\right\}_{\alpha=\alpha_0}. \tag{35}$$

Equating coefficients of $(iK\beta^2)^p$ we get

$$\frac{8\Omega_0\alpha_0(-1)^{p+1}}{K} = \sum_n^{\infty} \frac{2^p(n+1)\,!}{p\,(p+1)(n-p)\,!}\,\chi_{2n}, \tag{36}$$

In this relation if we vary p from zero to infinity we get an infinite set of equations in χ_{2n} where χ_{2n} are infinite sets of constants. The number of unknowns in each successive equation is always less by one. These equations can be solved to any desired degree of accuracy. For χ_{2n} we get the relation of the type

$$\left\{A_n\frac{\partial}{\partial\alpha}W_{n+1,\,\frac{1}{2}}(2iK\alpha^2)+B_n\frac{\partial}{\partial\alpha}W_{-n-1,\,\frac{1}{2}}(-2iK\alpha^2)\right\}_{\alpha=\alpha_0}=\int_n^{(2)}$$

where $\int_n^{(2)}$ is a known constant depending upon n. Constants A_n and B_n can now be determined by solving equations of the type (34) and (36). Using these values expressions for the velocity components can be determined from (34) and (26),

In conclusion I thank Dr. S. D. Nigam for his kind interest in the preparation of this paper,

References

Taylor. G. I (1922) *Proc Roy. Soc., Ser. A*, **102**, 180.

Long, R. R. (1952), *Technical Report No. 1*, The John Hopkins University.

Nigam and Chatterji (1953) Unpublished. Motion of a symmetrical body along the axis of a rotating liquid

Fadnis. B, (1955), *Bull. Cal. Math, Soc.*, **47**, 143.

Goldstein (1952), Modern Developments in Fluid Dynamics, Oxford University Press. Chapter III.

Bateman (1937) *Partial Differential Equations of Physics* Chapter on Paraboidal coordinates.